Section 16.3 Part III: Quotient rings

Recall the following lemma about cosets.

Lemma 1. Let H be an additive subgroup of the group $(G, +)$. Then the following are equivalent:

(1) $a + H = b + H$

(2) $b \in a + H$

(3) $b - a \in H$

1 Multiplication of cosets is well-defined

Let R be a ring, let $s, s', t, t' \in R$, and let I be an ideal of R. Suppose also that

$$s' \in s + I \text{ and } t' \in t + I.$$

Show that

$$s't' \in st + I \tag{1}$$

Solution: Since $s' \in s + I$, we have $s' = s + a$ for some $a \in I$. Similarly, since $t' \in t + I$, we have $t' = t + b$ for some $b \in I$. Then

$$s't' = (s + a)(t + b) = st + (sb + at + ab)$$

We have $sb, at, ab \in I$ because I is an ideal (and therefore satisfies the “absorbing” property). Therefore $sb+at+ab \in I$ since I is a subring (and therefore is closed under the ring multiplication operation for R). So $s't' \in st + I$.

(Note that (1) implies $s't' + I = st + I$, due to Lemma 1)

2 The zero element in a quotient ring

What is the zero element in a quotient ring R/I?

Solution: The zero element in the quotient ring R/I is the identity element of the additive quotient group $(R/I, +)$. The identity element in the quotient group $(R/I, +)$ is the subgroup I. So $\boxed{\text{the zero element in the quotient ring } R/I \text{ is the ideal } I\text{, thought of as a coset in } R/I}$.

3 Example: matrices with integer entries

Consider the set

$$\mathrm{Mat}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

of 2×2 matrices with integer entries, under the usual matrix addition and matrix multiplication.

Let I be the subset of $\mathrm{Mat}_2(\mathbb{Z})$ consisting of matrices with even entries. You have shown in previous homework/quizzes that I is an ideal of $\mathrm{Mat}_2(\mathbb{Z})$.

(a) Consider the coset

$$\begin{pmatrix} 7 & 8 \\ 5 & -3 \end{pmatrix} + I$$

and the coset

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + I.$$

Are they the same element in the quotient ring $\mathrm{Mat}_2(\mathbb{Z})/I$? Explain.

Solution: Yes

(b) Are the cosets $\begin{pmatrix} 2 & 8 \\ 5 & -3 \end{pmatrix} + I$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + I$ the same set? Explain.

Solution: No.

(c) What are the elements (the cosets) in the quotient ring $\mathrm{Mat}_2(\mathbb{Z})/I$? How many are there?

Solution: We have $\mathrm{Mat}_2(\mathbb{Z})/I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} + I : a, b, c, d \in \{0, 1\} \right\}$. It consists of $16 = 2^4$ cosets.

(d) What is the zero element in the quotient ring $\mathrm{Mat}_2(\mathbb{Z})/I$? Describe this set of matrices.

Solution: Earlier, we said that $\boxed{\text{the zero element in the quotient ring } R/I \text{ is the ideal } I}$. So the zero element in this case is I, the set of 2×2 matrices with even entries.

(e) The quotient ring $\mathrm{Mat}_2(\mathbb{Z})/I$ has unity. What is the unity element (it is a coset)? Describe the elements (matrices) in this coset.

Solution: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \text{ are odd integers}, b, c \text{ are even integers}\} \right\}$

Section 16.3 Part IV: First isomorphism theorem

4 Example: The canonical surjective homomorphism

Let R denote the ring

$$\mathrm{Mat}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

of 2×2 integer matrices, and let I denote the ideal consisting of matrices with even entries.

(a) Write down the canonical surjective homomorphism π from $\mathrm{Mat}_2(\mathbb{Z})$ onto $\mathrm{Mat}_2(\mathbb{Z})/I$ (from R onto R/I).

(b) What is the kernel of π?

(c) What is the image of π?

(d) Is π an injective homomorphism?

(e) Is π an isomorphism?

Solution: (See Theorem 16.30 in Judson Section 16.3)

- The canonical map is the map $\pi : \mathrm{Mat}_2(\mathbb{Z}) \to \mathrm{Mat}_2(\mathbb{Z})/I$ defined by

$$x \mapsto x + I$$

- The zero element of the quotient ring $\mathrm{Mat}_2(\mathbb{Z})/I$ is the coset I, so the kernel is the set of matrices in I.
- The image is the entire codomain (since this map is surjective).
- This map is not injective (and hence not an isomorphism), since the kernel is I which contains more than just the zero element of the domain.

5 Example: First isomorphism theorem

Let R be the ring defined in the previous question. Let S denote

$$\mathrm{Mat}_2(\mathbb{Z}_2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_2 = \{0, 1\} \right\}$$

of 2×2 matrices with entries in $Z_2 = \{0, 1\}$.

(a) Write a few of the elements (the matrices) in the ring $S = \mathrm{Mat}_2(\mathbb{Z}_2)$? How many are there?

Solution: There are $16 = 2^4$ matrices.

(b) Write down the most natural ring homomorphism $f : R \to S$ you can think of which is surjective.

(c) Prove that your f is a ring homomorphism (that is, it preserves the ring addition and ring multiplication).

(d) What is the kernel of your f?

(e) Write f as the composition of two ring homomorphisms

$$f = \iota \cdot \pi$$

such that π is surjective and ι is injective. Draw a commutative diagram to visualize this composition.

(f) Apply the First isomorphism theorem (for rings) to this situation.

Solution: Let R, S, I be the specific rings defined in this question and the previous question. Then the quotient ring R/I is isomorphic to S.