1 Isomorphism

Consider the map $\varphi : \mathbb{C} \to \mathbb{C}$ defined by

$$\varphi(a + bi) = a - bi$$

(i) Prove that φ preserves addition.

Solution: Suppose $x = a + bi$, $y = c + di$, where $a, b, c, d \in \mathbb{R}$. Then

$$\begin{aligned}
\varphi(x + y) &= \varphi(a + bi + c + di) \\
&= \varphi((a + c) + (b + d)i) \\
&= (a + c) - (b + d)i \\
&= (a - bi) + (c - di) \\
&= \varphi(a + bi) + \varphi(c + di) \\
&= \varphi(x) + \varphi(y).
\end{aligned}$$

(ii) Prove that φ is surjective.

Solution: Suppose $z \in \mathbb{C}$. Then $z = a + bi$ for some $a, b \in \mathbb{R}$. Then $a - bi \in \mathbb{C}$ such that $\varphi(a - bi) = a + bi = z$.

2 Evaluation homomorphism

Let $\mathbb{Z}[x]$ denote the ring of all polynomials having integer coefficients.
Consider the evaluation homomorphism $\mathrm{ev} : \mathbb{Z}[x] \to \mathbb{R}$ defined by

$$p(x) \mapsto p(2)$$

(a) What is the kernel of ev?

Solution: The set of integer polynomials with root 2.

(b) What is the image of ev?

Solution: The ring of integers $\mathbb{Z}$.

3 Matrices with integer entries

Definition 1. Consider the set

$$\text{Mat}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

of 2×2 matrices with integer entries. It forms a ring with unity $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ under the usual matrix addition and matrix multiplication. The zero element is the zero matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Let I be the subset of $\text{Mat}_2(\mathbb{Z})$ consisting of matrices with even entries. Prove that

$$I \text{ is an ideal of } \text{Mat}_2(\mathbb{Z}).$$

(You need to show that:

- I is an additive subgroup of $\text{Mat}_2(\mathbb{Z})$
- I "absorbs" all elements of $\text{Mat}_2(\mathbb{Z})$, that is, for all $a \in I$ and $r \in \text{Mat}_2(\mathbb{Z})$, we have $ar \in I$ and $ra \in I$.)

4 An ideal of the ring of integer polynomials?

Let $\mathbb{Z}[x]$ denote the ring of all polynomials having integer coefficients. Consider the subset T of $\mathbb{Z}[x]$ of polynomials $f(x)$ such that $f(0) = 5$.

(a) What do the polynomials in S look like? Give some examples.

(b) Is T an ideal?

> **Solution:** No, it's not even a subring. For example, $f(x) = x + 5$ is in T, but the product $f(x)f(x) = x^2 + 10x + 25$ is not in T.

(c) If T is a principal ideal, describe an element of T which generates T.

> **Solution:** T is not an ideal.

5 Ideals?

Let $\mathbb{Z}[x]$ denote the ring of all polynomials having integer coefficients. Which of the following subsets of $\mathbb{Z}[x]$ are ideals? Answer **Yes** or **No**.

- If you answer No, provide a specific example of how the subset fails the absorbing property of an ideal or how the subset fails to be an additive subgroup of $\mathbb{Z}[x]$.
- If you answer Yes, explain why the absorbing property holds (you don't need to prove that the subset is an additive group).

(a) S is the set consisting of the constant zero function and of all polynomials with no constant term.

Solution: Yes, S can be written in set-builder notation as $\{f(x)x : x \in \mathbb{Z}[x]\}$, which shows that S is the principal ideal generated by the polynomial x. Notation: $\langle x \rangle$.

Note that we don't need to do the "ideal test" because we see that S is a principal ideal (and therefore an ideal).

(b) $S = \mathbb{Z}$, that is, all the constant polynomials in $\mathbb{Z}[x]$.

Solution: S is not an ideal because it fails the absorbing property. For example, $f(x) = 5$ is a polynomial in S and $g(x) = x^2$ is a polynomial in $\mathbb{Z}[x]$, but their product is $5x^2$ which is not in S.

(c) The set S of integer polynomials $f(x)$ such that $f(5) \neq 0$, i.e. 5 is not a root of $f(x)$.

Solution: S is not an ideal because it fails the absorbing property. For example, $f(x) = x^2$ is a polynomial in S and $g(x) = (x - 5)$ is a polynomial in $\mathbb{Z}[x]$, but their product is $(x - 5)x^2$ which is not in S.

(d) The set S of integer polynomials $f(x)$ such that $f'(2) = 0$, i.e. 2 is a root of $f'(x)$.

Solution: S is not an ideal because it fails the "absorbing" property. For example, $f(x) = x^2 - 4x$ is a polynomial in S (since $f'(x) = 2x - 4$), and $g(x) = x$ is a polynomial in $\mathbb{Z}[x]$, however, their product is $x^3 - 4x^2$ which has derivative $3x^2 - 4x$ which is not in S.

6 An ideal of the ring of integers

Consider the subset

$$n\mathbb{Z} = \{nk : k \in \mathbb{Z}\} = \{..., -2n, -n, 0, n, 2n, ...\}$$

of the ring $\mathbb{Z}$ of integers.

Is $n\mathbb{Z}$ an ideal? Is $n\mathbb{Z}$ a principal ideal? If it is, describe an element of $n\mathbb{Z}$ which generates $n\mathbb{Z}$

Hint: See Example 16.26 in Judson Section 16.3 Ring homomorphisms and ideals

Solution: $n\mathbb{Z} = \langle n \rangle$ is the principal ideal generated by n.