Reference: Week 10 class notes, Textbook's Section 16.1 and Textbook's Section 16.2)

1 Definitions

Write down the definition of ...

- zero element
- unity (or identity)
- ring
- commutative ring
- ring with unity (or ring with identity)
- integral domain
- field
- zero divisor
- unit

2 Gaussian integers

(a) Write down the definition of the set $\mathbb{Z}[i]$ of the Gaussian integers.

(b) Is $\mathbb{Z}[i]$ a subring of the ring of complex numbers (under usual addition and multiplication)?

> **Solution:** Yes. All four properties of being a subring are satisfied.

(c) Is $\mathbb{Z}[i]$ a commutative ring?

> **Solution:** Yes because $\mathbb{C}$ is a commutative ring.

(d) Is $\mathbb{Z}[i]$ an integral domain?

> **Solution:** See class notes or Example 16.12 Textbook's Section 16.2

(e) What are the units of $\mathbb{Z}[i]$?

> **Solution:** See class notes or Example 16.12 Textbook's Section 16.2

(f) Is $\mathbb{Z}[i]$ a field?

> **Solution:** No because not all nonzero elements are units. (A field is a commutative ring with unity such that every nonzero elements are units.)

3 Question

Definition 1. Consider the set

$$\mathrm{Mat}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

of 2×2 matrices with integer entries. It forms a (non-commutative) ring with unity under the usual matrix addition and matrix multiplication. The unity of $\mathrm{Mat}_2(\mathbb{Z})$ is the identity matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the zero element is the zero matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

For each of the following subsets S of $\mathrm{Mat}_2(\mathbb{Z})$, answer whether S is a subring of $\mathrm{Mat}_2(\mathbb{Z})$. (Answer **Yes/ No**)

If you claim S is not a subring, specify which subring conditions are not satisfied (S doesn't contain the zero element; S is not closed under ring addition; S is not closed under ring negation; S is not closed under ring multiplication)

(a) S is the subset of $\mathrm{Mat}_2(\mathbb{Z})$ consisting of invertible matrices.

> **Solution:** No, S is not a subring. It is closed under ring multiplication, but it fails the other three properties: S doesn't contain the zero element; S is not closed under ring addition; S is not closed under ring negation.

(b) $S = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : a, c, d \in \mathbb{Z} \right\}$ is the subset of lower-triangular matrices in $\mathrm{Mat}_2(\mathbb{Z})$.

> **Solution:** Yes, S is a subring. All four properties are satisfied.

(c) $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ is the subset of diagonal matrices in $\mathrm{Mat}_2(\mathbb{Z})$.

> **Solution:** Yes, S is a subring. All four properties are satisfied.

4 Question

Definition 2. Let R be a ring which has unity denoted by the symbol $\mathbf{1}$. An element $u \in R$ is called a *unit* (also called an *invertible element*) if there exists $v \in R$ such that $uv = vu = \mathbf{1}$.

What are the units (if any) in the ring $\mathbb{Z}_{10}$?

(Hint: Example 3.11 in Section 3.2 Groups: Definitions and Examples computes the units for $\mathbb{Z}_8$.)

Solution: The units are the nonzero elements which are relatively prime to 10:

$$1, 3, 7, 9$$

A notation that we have used all semester for this set of units is $U(10)$.

5 Fields

For this question, use the definition and choose examples from class notes or Textbook's Section 16.2 or Section 16.1.

(a) Give an example of an infinite field.

Solution: $\mathbb{Q}, \mathbb{R}, \mathbb{C}$

(b) Give an example of a finite field.

Solution: See Example 16.17 Textbook's Section 16.2. For each prime p, the ring $\mathbb{Z}_p$ is a field. See also Example 16.13.

6 Question

Suppose R is a ring with unity $\mathbf{1}$. Prove the following: if $x^4 = \mathbf{0}$ then $\mathbf{1} - x$ is a unit.

Solution: Let $v = \mathbf{1} + x + x^2 + x^3$. Then we have

$$\begin{aligned}(\mathbf{1} - x)v &= (\mathbf{1} - x)(\mathbf{1} + x + x^2 + x^3) \\ &= \mathbf{1} + x + x^2 + x^3 - x(\mathbf{1} + x + x^2 + x^3) \\ &= \mathbf{1} - x^4 \\ &= \mathbf{1} \text{ since } x^4 = \mathbf{0}\end{aligned}$$

7 Question

(a) Write down the definition of a *zero divisor*. (Use class notes or Textbook's Section 16.2 or Section 16.1)

(b) What are the zero divisors (if any) of the ring $\mathbb{Z}_{10}$?

Solution: All the nonzero elements which are not units, $\boxed{2,4,5,6,8}$, are zero divisors. For example, 4 is a zero divisor because $(4)(5) = 0$.

8 Question

Definition 3. Let R be a ring. An element x in R is called an *idempotent* if it satisfies $x^2 = x$.

What are the idempotents in $\mathbb{Z}_6$? (Hint: For each of the elements r in $\mathbb{Z}_6$, simply check whether $r^2 = r$.)

Solution: The idempotents in $\mathbb{Z}_6$ are $\boxed{0, 1, 3, 4}$. The computation is below:

0 is an idempotent, since $0^2 = 0$

1 is an idempotent, since $1^2 = 1$

2 is not an idempotent, since $2^2 = 4 \neq 2$

3 is an idempotent, since $3^2 = 3$

4 is an idempotent, since $4^2 = 4$

5 is not an idempotent, since $5^2 = 1 \neq 5$

9 Question

(a) Write down the definition of the *characteristic* of a ring. (Use class notes or Textbook's Section 16.2)

(b) Write down the statement and proof of Lemma 16.18 from Textbook's Section 16.2

(c) What is the characteristic of the ring $\mathbb{R}$ of real numbers?

Solution: The order of the unity element 1 is infinite, so the characteristic of the ring is 0.

(d) What is the characteristic of the ring $\mathbb{Z}_6$?

Solution: The order of the unity element 1 is 6, so the characteristic of $\mathbb{Z}_6$ is 6.