Reference: Week 10 class notes, Textbook's Section 16.1 and Textbook's Section 16.2)

1 Definitions

Write down the definition of ...

- zero element
- unity (or identity)
- ring
- commutative ring
- ring with unity (or ring with identity)
- integral domain
- field
- zero divisor
- unit

2 Gaussian integers

(a) Write down the definition of the set $\mathbb{Z}[i]$ of the Gaussian integers.

(b) Is $\mathbb{Z}[i]$ a subring of the ring of complex numbers (under usual addition and multiplication)?

(c) Is $\mathbb{Z}[i]$ a commutative ring?

(d) Is $\mathbb{Z}[i]$ an integral domain?

(e) What are the units of $\mathbb{Z}[i]$?

(f) Is $\mathbb{Z}[i]$ a field?

3 Question

Definition 1. Consider the set

$$\mathrm{Mat}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

of 2×2 matrices with integer entries. It forms a (non-commutative) ring with unity under the usual matrix addition and matrix multiplication. The unity of $\mathrm{Mat}_2(\mathbb{Z})$ is the identity matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the zero element is the zero matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

For each of the following subsets S of $\mathrm{Mat}_2(\mathbb{Z})$, answer whether S is a subring of $\mathrm{Mat}_2(\mathbb{Z})$. (Answer **Yes/ No**)

If you claim S is not a subring, specify which subring conditions are not satisfied (S doesn't contain the zero element; S is not closed under ring addition; S is not closed under ring negation; S is not closed under ring multiplication)

(a) S is the subset of $\mathrm{Mat}_2(\mathbb{Z})$ consisting of invertible matrices.

(b) $S = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : a, c, d \in \mathbb{Z} \right\}$ is the subset of lower-triangular matrices in $\mathrm{Mat}_2(\mathbb{Z})$.

(c) $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ is the subset of diagonal matrices in $\mathrm{Mat}_2(\mathbb{Z})$.

4 Question

Definition 2. Let R be a ring which has unity denoted by the symbol $\mathbf{1}$. An element $u \in R$ is called a *unit* (also called an *invertible element*) if there exists $v \in R$ such that $uv = vu = \mathbf{1}$.

What are the units (if any) in the ring $\mathbb{Z}_{10}$?

(Hint: Example 3.11 in Section 3.2 Groups: Definitions and Examples computes the units for $\mathbb{Z}_8$.)

5 Fields

For this question, use the definition and choose examples from class notes or Textbook's Section 16.2 or Section 16.1.

(a) Give an example of an infinite field.

(b) Give an example of a finite field.

6 Question

Suppose R is a ring with unity $\mathbf{1}$. Prove the following: if $x^4 = \mathbf{0}$ then $\mathbf{1} - x$ is a unit.

7 Question

(a) Write down the definition of a *zero divisor*. (Use class notes or Textbook's Section 16.2 or Section 16.1)

(b) What are the zero divisors (if any) of the ring $\mathbb{Z}_{10}$?

8 Question

Definition 3. Let R be a ring. An element x in R is called an *idempotent* if it satisfies $x^2 = x$.

What are the idempotents in $\mathbb{Z}_6$? (Hint: For each of the elements r in $\mathbb{Z}_6$, simply check whether $r^2 = r$.)

9 Question

(a) Write down the definition of the *characteristic* of a ring. (Use class notes or Textbook's Section 16.2)

(b) Write down the statement and proof of Lemma 16.18 from Textbook's Section 16.2

(c) What is the characteristic of the ring $\mathbb{R}$ of real numbers?

(d) What is the characteristic of the ring $\mathbb{Z}_6$?