1 First isomorphism theorem (statement)

Let $f : G \to H$ be a homomorphism of groups. What does the first isomorphism theorem say?

2 First isomorphism theorem (example)

Let D_4 be the dihedral group of order 8

$$\begin{aligned} D_4 =&\{e, R, R^2, R^3, \\ &f, fR, fR^2, fR^3\} \end{aligned}$$

and let

$$V_4 = \{Id, h, v, r\}$$

be the (non-square) rectangle mattress group.
Consider the homomorphism $\phi : D_4 \to V_4$ where $\phi(R) = h$ and $\phi(f) = v$.

(a) Using the homomorphism property $\phi(ab) = \phi(a)\phi(b)$, find where ϕ sends all elements of D_4.

> **Solution:** For example,
>
> $$\phi(R^3) = \phi(R)\phi(R)\phi(R) = hhh = eh = h,$$
>
> and
>
> $$\phi(fR) = \phi(f)\phi(R) = vh = r.$$
>
> Compute where ϕ sends the rest of the elements.

(b) Find $\ker(\phi)$.

(c) Is ϕ injective? If not, state whether it is a 2-to-1 or 4-to-1 or 8-to-1 mapping.

> **Solution:** No, $\ker(\phi) = \{e, R^2\}$ has cardinality 2, so ϕ is a two-to-one mapping.

(d) Pick a coset of $\ker(\phi)$ not equal to $\ker(\phi)$, for example, $R\ker(\phi)$ or $f\ker(\phi)$. Write down all elements of this coset, and then demonstrate that ϕ sends all elements of this coset to the same element in the codomain V_4. Write this coset as the fiber of an element in the image of ϕ.

> **Solution:** For example, the coset $R\ker(\phi)$ is equal to $\{R, R^3\}$. Both elements in this coset are sent to h by ϕ. We can write the coset $R\ker(\phi)$ as the fiber $\phi^{-1}(\{h\})$.

(e) Find $\text{Im}(\phi)$.

(f) What does the first isomorphism theorem tells us about ϕ?

Solution: $D_4/\langle R^2 \rangle$ is isomorphic to $\text{Im}(\phi) = V_4$.

3 First isomorphism theorem (conceptual)

(a) Given a homomorphism $f : G \to H$, how can you construct a normal subgroup of G using f?

(b) Given a normal subgroup N of G, how can you construct a homomorphism whose domain is G and whose kernel is N?

Solution: The natural (or canonical) homomorphism $f : G \to G/N$ given by $x \mapsto xN$ has N as its kernel.

4 Fundamental Theorem of Finite Abelian Groups

Apply the Fundamental Theorem of Finite Abelian Groups (Judson Section 13.1) to answer these.

(a) True or false? The group D_{12} is isomorphic to the group $\mathbb{Z}_3 \times \mathbb{Z}_4$

Solution: False. The dihedral group D_{12} is not abelian.

(b) Which nontrivial direct product is $\mathbb{Z}_{12}$ isomorphic to?

Solution: $\mathbb{Z}_4 \times \mathbb{Z}_3$

(c) True or false? The group $\mathbb{Z}_{14}$ is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_7$

Solution: True because the $gcd(2, 7) = 1$. Alternatively, you can use $(1, 1) \in \mathbb{Z}_2 \times \mathbb{Z}_7$ to generate the entire group and thus showing that it is a cyclic group of order 14.

(d) True or false? The group $\mathbb{Z}_{16}$ is isomorphic to the group $\mathbb{Z}_4 \times \mathbb{Z}_4$

Solution: False because the $gcd(4, 4) = 4$. See week 9 class notes on "Fundamental theorem of finite abelian groups".

Alternatively, you can check that every element in the group $\mathbb{Z}_4$ has order 1, 2, or 4, so every element in $\mathbb{Z}_4 \times \mathbb{Z}_4$ also has order 1, 2, or 4, and thus no element can generate the entire group.

5 A subgroup

Prove that the subset

$$Z(G) = \{a \in G : ax = xa \text{ for all } x \in G\}$$

of G is a subgroup of G.

Solution:

Proof. To show that $Z(G)$ is a subgroup, we need to show the following: (1) The identity e of G is contained in $Z(G)$; (2) the subset $Z(G)$ is closed under the group operation of G; and (3) the subset $Z(G)$ is closed under taking inverses.

(1) The identity e is in $Z(G)$ because $ex = e = xe$ for all $x \in G$ by definition of the identity element.

(2) To show that the subset $Z(G)$ is closed under the group operation, we need to show that if $a, b \in Z(G)$ then $ab \in Z(G)$.

Suppose $a, b \in Z(G)$. (Our goal is to show that $ab \in Z(G)$, that is, $(ab)x = x(ab)$ for all $x \in G$.) Let $x \in G$. Then we have

$$\begin{aligned} (ab)x &= a(bx) \\ &= a(xb) \text{ since } b \text{ is in } Z(G) \\ &= (ax)b \\ &= (xa)b \text{ since } a \text{ is in } Z(G) \\ &= x(ab) \end{aligned}$$

This concludes the proof that $ab \in Z(G)$.

(3) To show that the subset $Z(G)$ is closed under taking inverses, we need to show that if $a \in Z(G)$ then its inverse a^{-1} is also in $Z(G)$.

Suppose $a \in Z(G)$. (Our goal is to show that $a^{-1} \in Z(G)$), that is, $a^{-1}x = xa^{-1}$ for all $x \in G$.) Let $x \in G$. Then we have

$$xa = ax,$$

since $a \in Z(G)$. Multiply on the left and on the right by a^{-1}:

$$\begin{aligned} a^{-1}(xa)a^{-1} &= a^{-1}(ax)a^{-1} \\ (a^{-1}x)(aa^{-1}) &= (a^{-1}a)(xa^{-1}) \\ (a^{-1}x)e &= e(xa^{-1}) \\ a^{-1}x &= xa^{-1}, \end{aligned}$$

as needed.

□

6 Product of subgroups

Suppose G is a group, and $H \leq G$ and $N \trianglelefteq G$. Prove that HN is a subgroup of G.

Solution: HW 09

7 Intersection of subgroups

Suppose that G is a group, and $H \leq G$ and $N \trianglelefteq G$. Prove that $H \cap N$ is a *normal* subgroup of H.

Solution: HW 09

8 Computation

Let $G = S_4$, $H = \langle\ (1234)\ \rangle$, and $N = A_4$. Compute H, N, and $H \cap N$. Then use the fact that

$$H/(H \cap N) \cong (HN)/N$$

to quickly compute HN. What is HN equal to?

Solution: HW 09

9 The First Isomorphism Theorem in words

Explain the first isomorphism theorem in words to a classmate who has not seen it before. Do not use any math symbol. You might enjoy reading the blog post

The First Isomorphism Theorem, Intuitively by Tai-Danae Bradley (Math3ma).

10 Counting abelian groups

How many abelian groups of order $540 = 2^2 \cdot 3^3 \cdot 5$ are there, up to isomorphism? List all of them.

Solution: There are six isomorphism classes. They are given in Judson Example 13.5 for an example on how to use the Fundamental Theorem of Finite Abelian Groups to count isomorphism classes of abelian groups.

11 First isomorphism theorem (another example)

Consider the homomorphism $f : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by

$$f(x) = 3x.$$

(a) What is $\ker f$? Is f injective? f is a t-to-one mapping for some positive integer. What is t?

Solution: f is a 3-to-one mapping, since $\ker f = \{0, 4, 8\}$ is of order 3. It's not injective.

(b) Find all cosets of $\ker f$.

> **Solution:** Since $\mathbb{Z}_{12}$ has 12 elements and $\ker f$ has three, there should be four cosets.
>
> Let K denote $\ker f$. Two of the cosets are K itself and $2 + K = \{2, 6, 10\}$. Find the other two cosets.

(c) Consider the coset $2 + \ker f$. Find all elements of this coset.

> **Solution:** The coset $2 + \ker f$ is equal to $\{2, 6, 10\}$

(d) Find all elements in the fiber $f^{-1}(\{6\})$.

(e) True or false? The fiber $f^{-1}(\{6\})$ is a coset of $\ker f$.

> **Solution:** True. $f^{-1}(\{6\} = \{2, 6, 10\}$, which is the coset $2 + \ker f$.
>
> Note: In fact, we saw in class that every fiber $f^{-1}(\{y\})$ is a coset of $\ker f$, for every $y \in \text{Im} f$.

(f) What is $\text{Im} f$? What does the first isomorphism theorem tells us about ϕ?

> **Solution:** $\text{Im} f$=$\{3, 6, 9, 0\}$ The quotient group $\mathbb{Z}_{12}/\{0, 4, 8\}$ is isomorphic to $\{3, 6, 9, 0\}$, which is a cyclic group of order 4.

12 First isomorphism theorem (yet another example)

Recall that $\mathbb{R}^*$ denotes the set of nonzero real numbers equipped with usual multiplication as the binary operation. Consider the homomorphism $f : \mathbb{R}^* \to \mathbb{R}^*$ defined by

$$f : x \mapsto x^2$$

(a) What is $\ker f$?

> **Solution:** $\{1, -1\}$

(b) Is f injective? If not, state whether it is a 2-to-1, or 3-to-1, or 4-to-1 mapping, etc.

(c) Pick a coset of $\ker(f)$ not equal to $\ker(f)$, for example, $2 \ker(\phi)$ or $(-3) \ker(\phi)$. Write down all elements of this coset, and then demonstrate that f sends all elements of this coset to the same element in the image of f. Write this coset as the fiber of an element in $\text{Im} f$.

> **Solution:** Let K denote the kernel of f. For example, the coset $(-3)K$ is equal to $\{-3, 3\}$. Both elements in this coset are sent to 9 by f. So we can write the coset $(-3)K$ as the fiber $f^{-1}(\{9\})$.

(d) What is $\mathrm{Im} f$?

Solution: $\mathbb{R}^*_{>0}$, the subgroup of $\mathbb{R}^*$ consisting of all positive real numbers.

(e) What does the first isomorphism theorem tells us about ϕ?

Solution: The quotient group $\mathbb{R}^*/\{1, -1\}$ is isomorphic to $\mathbb{R}^*_{>0}$.

13 First isomorphism theorem (last example)

Recall that $\mathbb{C}^*$ denotes the set of nonzero complex numbers equipped with usual multiplication as the binary operation. Consider the homomorphism $f : \mathbb{C}^* \to \mathbb{R}^*$ defined by

$$f : z \mapsto |z|$$

Since $|a + bi| = \sqrt{a^2 + b^2}$, this is the same as saying that $f(a + bi) = \sqrt{a^2 + b^2}$ for all $a + bi \in \mathbb{C}^*$
You can use the fact that $|xy| = |x||y|$ for all complex numbers x, y.

(a) What is $\ker f$?

Solution: $\{z \in \mathbb{C}^* : |z| = 1\}$, that is, the circle group $\mathbb{T}$.

(b) Is f injective? If not, state whether it is a 2-to-1, or 3-to-1, or 4-to-1 mapping, etc.

(c) Pick a coset of $\ker(f)$ not equal to $\ker(f)$, for example, $2\ker(\phi)$ or $3i\ker(\phi)$. Write this coset as the fiber of an element in $\mathrm{Im} f$.

Solution: Let K denote the kernel of f. For example, the coset $3iK$ is equal to $\{3iz : |z| = 1\}$. So every element $3iz$ in this coset is sent to $f(3iz) = |3||i||z| = 3 \cdot 1 \cdot 1 = 3$. So $\boxed{3iK = f^{-1}(\{3\})}$.

(d) What is $\mathrm{Im} f$?

Solution: $\mathbb{R}^*_{>0}$, the subgroup of $\mathbb{R}^*$ consisting of all positive real numbers.

(e) What does the first isomorphism theorem tell us?

Solution: The quotient group $\mathbb{C}^*/\mathbb{T}$ is isomorphic to $\mathbb{R}^*_{>0}$.