# 1 First isomorphism theorem (statement)

Let $f : G \to H$ be a homomorphism of groups. What does the first isomorphism theorem say?

# 2 First isomorphism theorem (example)

Let $D_4$ be the dihedral group of order 8

$$D_4 = \{e, R, R^2, R^3, \\ f, fR, fR^2, fR^3\}$$

and let

$$V_4 = \{Id, h, v, r\}$$

be the (non-square) rectangle mattress group.
Consider the homomorphism $\phi : D_4 \to V_4$ where $\phi(R) = h$ and $\phi(f) = v$.

(a) Using the homomorphism property $\phi(ab) = \phi(a)\phi(b)$, find where $\phi$ sends all elements of $D_4$.

(b) Find $\ker(\phi)$.

(c) Is $\phi$ injective? If not, state whether it is a 2-to-1 or 4-to-1 or 8-to-1 mapping.

(d) Pick a coset of $\ker(\phi)$ not equal to $\ker(\phi)$, for example, $R\ker(\phi)$ or $f\ker(\phi)$. Write down all elements of this coset, and then demonstrate that $\phi$ sends all elements of this coset to the same element in the codomain $V_4$. Write this coset as the fiber of an element in the image of $\phi$.

(e) Find $\text{Im}(\phi)$.

(f) What does the first isomorphism theorem tells us about $\phi$?

# 3 First isomorphism theorem (conceptual)

(a) Given a homomorphism $f : G \to H$, how can you construct a normal subgroup of $G$ using $f$?

(b) Given a normal subgroup $N$ of $G$, how can you construct a homomorphism whose domain is $G$ and whose kernel is $N$?

# 4 Fundamental Theorem of Finite Abelian Groups

Apply the Fundamental Theorem of Finite Abelian Groups (Judson Section 13.1) to answer these.

(a) True or false? The group $D_{12}$ is isomorphic to the group $\mathbb{Z}_3 \times \mathbb{Z}_4$

(b) Which nontrivial direct product is $\mathbb{Z}_{12}$ isomorphic to?

(c) True or false? The group $\mathbb{Z}_{14}$ is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_7$

(d) True or false? The group $\mathbb{Z}_{16}$ is isomorphic to the group $\mathbb{Z}_4 \times \mathbb{Z}_4$

# 5 A subgroup

Prove that the subset

$$Z(G) = \{a \in G : ax = xa \text{ for all } x \in G\}$$

of $G$ is a subgroup of $G$.

# 6 Product of subgroups

Suppose $G$ is a group, and $H \leq G$ and $N \trianglelefteq G$. Prove that $HN$ is a subgroup of $G$.

# 7 Intersection of subgroups

Suppose that $G$ is a group, and $H \leq G$ and $N \trianglelefteq G$. Prove that $H \cap N$ is a *normal* subgroup of $H$.

# 8 Computation

Let $G = S_4$, $H = \langle\ (1234)\ \rangle$, and $N = A_4$. Compute $H$, $N$, and $H \cap N$. Then use the fact that

$$H/(H \cap N) \cong (HN)/N$$

to quickly compute $HN$. What is $HN$ equal to?

# 9 The First Isomorphism Theorem in words

Explain the first isomorphism theorem in words to a classmate who has not seen it before. Do not use any math symbol. You might enjoy reading the blog post

The First Isomorphism Theorem, Intuitively by Tai-Danae Bradley (Math3ma).

# 10 Counting abelian groups

How many abelian groups of order $540 = 2^2 \cdot 3^3 \cdot 5$ are there, up to isomorphism? List all of them.

# 11 First isomorphism theorem (another example)

Consider the homomorphism $f : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by

$$f(x) = 3x.$$

(a) What is $\ker f$? Is $f$ injective? $f$ is a $t$-to-one mapping for some positive integer. What is $t$?

(b) Find all cosets of $\ker f$.

(c) Consider the coset $2 + \ker f$. Find all elements of this coset.

(d) Find all elements in the fiber $f^{-1}(\{6\})$.

(e) True or false? The fiber $f^{-1}(\{6\})$ is a coset of $\ker f$.

(f) What is Im$f$? What does the first isomorphism theorem tells us about $\phi$?

# 12 First isomorphism theorem (yet another example)

Recall that $\mathbb{R}^*$ denotes the set of nonzero real numbers equipped with usual multiplication as the binary operation. Consider the homomorphism $f : \mathbb{R}^* \to \mathbb{R}^*$ defined by

$$f : x \mapsto x^2$$

(a) What is $\ker f$?

(b) Is $f$ injective? If not, state whether it is a 2-to-1, or 3-to-1, or 4-to-1 mapping, etc.

(c) Pick a coset of $\ker(f)$ not equal to $\ker(f)$, for example, $2\ker(\phi)$ or $(-3)\ker(\phi)$. Write down all elements of this coset, and then demonstrate that $f$ sends all elements of this coset to the same element in the image of $f$. Write this coset as the fiber of an element in Im$f$.

(d) What is Im$f$?

(e) What does the first isomorphism theorem tells us about $\phi$?

# 13 First isomorphism theorem (last example)

Recall that $\mathbb{C}^*$ denotes the set of nonzero complex numbers equipped with usual multiplication as the binary operation. Consider the homomorphism $f : \mathbb{C}^* \to \mathbb{R}^*$ defined by

$$f : z \mapsto |z|$$

Since $|a + bi| = \sqrt{a^2 + b^2}$, this is the same as saying that $f(a + bi) = \sqrt{a^2 + b^2}$ for all $a + bi \in \mathbb{C}^*$ You can use the fact that $|xy| = |x||y|$ for all complex numbers $x$, $y$.

(a) What is $\ker f$?

(b) Is $f$ injective? If not, state whether it is a 2-to-1, or 3-to-1, or 4-to-1 mapping, etc.

(c) Pick a coset of $\ker(f)$ not equal to $\ker(f)$, for example, $2\ker(\phi)$ or $3i\ker(\phi)$. Write this coset as the fiber of an element in Im$f$.

(d) What is Im$f$?

(e) What does the first isomorphism theorem tell us?