1. Prove or disprove: The groups $4\mathbb{Z}$ and $5\mathbb{Z}$ are isomorphic.

> **Solution:** True.
>
> **Proof:** The group
> $$4\mathbb{Z} = \{4k : k \in \mathbb{Z}\} = \langle 4 \rangle$$
> is a cyclic group of infinite order, and
> $$5\mathbb{Z} = \{5k : k \in \mathbb{Z}\} = \langle 5 \rangle$$
> is also a cyclic group of infinite order.
>
> Every infinite cyclic group is isomorphic to $\mathbb{Z}$, so $4\mathbb{Z}$ and $5\mathbb{Z}$ are isomorphic.
>
> An explicit isomorphism $f : 4\mathbb{Z} \to 5\mathbb{Z}$ can be given as follows: For each $4k \in 4\mathbb{Z}$, let
> $$f(4k) = 5k.$$

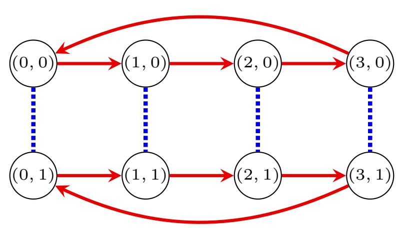
2. **Question:**

$$\text{If } H \text{ and } K \text{ are subgroups and } H \cong K, \text{ then are } G/H \text{ and } G/K \text{ isomorphic?}$$

To answer this question, consider the group $G := \mathbb{Z}_4 \times \mathbb{Z}_2$.

Below is the Cayley diagram for G with two generators,

$(1, 0)$ (of order 4, solid arrow) and $(0, 1)$ (of order 2, dotted edge):



We can visually demonstrate (using the above Cayley diagram) that the quotient of $\mathbb{Z}_4 \times \mathbb{Z}_2$ by the subgroup $H = \langle (0, 1) \rangle$ is the cyclic group $\mathbb{Z}_4$. To do this, collapse all the dotted edges representing H.

(a) List all elements (cosets) in the quotient group $G/H = (\mathbb{Z}_4 \times \mathbb{Z}_2)/\langle\ (0, 1)\ \rangle$ (or circle them).

> **Solution:** The quotient group $\mathbb{Z}_4 \times \mathbb{Z}_2/\langle (0, 1) \rangle$ consists of the cosets
> $$\langle (0, 1) \rangle, \qquad (1, 0) + \langle (0, 1) \rangle, \qquad (2, 0) + \langle (0, 1) \rangle, \qquad (3, 0) + \langle (0, 1) \rangle$$

(b) The quotient of $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ by the subgroup $K = \langle (2,0) \rangle$ is a harder to see. List all elements in the quotient group G/K.

Solution: The quotient group $G/K = \mathbb{Z}_4 \times \mathbb{Z}_2 / \langle (2,0) \rangle$ consists of the cosets

$$\langle (2,0) \rangle, \qquad (1,0) + \langle (2,0) \rangle, \qquad (0,1) + \langle (2,0) \rangle, \qquad (1,1) + \langle (2,0) \rangle.$$

(c) What familiar group is $G/K = (\mathbb{Z}_4 \times \mathbb{Z}_2)/\langle (2,0) \rangle$ isomorphic to?

Solution: This group is of order 4, so it's either isomorphic to $\mathbb{Z}_4$ or V_4. We can compute the order of all the non-identity elements:

$$((1,0) + \langle (2,0) \rangle) + ((1,0) + \langle (2,0) \rangle) = (2,0) + \langle (2,0) \rangle = \langle (2,0) \rangle,$$

so $(1,0) + \langle (2,0) \rangle$ has order 2.
Similarly, both

$$(0,1) + \langle (2,0) \rangle,$$
$$(1,1) + \langle (2,0) \rangle.$$

have order 2.
So no element generates the entire group, so it's not a cyclic group. Therefore, this group is isomorphic to V_4.

3. The *center* of a group G is the set

$$Z(G) = \{z \in G \mid gz = zg, \text{ for all } g \in G\} = \{z \in G \mid gzg^{-1} = z, \text{ for all } g \in G\}.$$

We proved in an earlier homework that $Z(G)$ is a subgroup of G. Now, prove that $Z(G)$ is normal in G.

4. Let $f : G \to H$ be a group homomorphism. Prove that the subgroup $\ker f$ is normal in G.

Solution: Let K denote $\ker f$. (We will show that $gkg^{-1} \in K$ for all $k \in K$ and $g \in G$.)
Let $k \in K$ and $g \in G$. We have

$$f(gkg^{-1}) = f(g)\, f(k)\, f(g^{-1}) = f(g)\, e\, f(g)^{-1} = e,$$

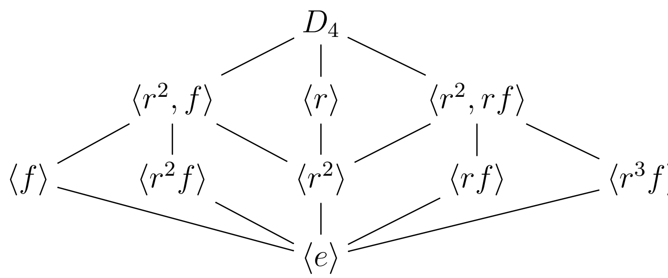
where the second equality is due to the fact that $k \in \ker f$ and the fact that the inverse of $f(g)$ is $f(g^{-1})$. Therefore, $gkg^{-1} \in K$.

5. Consider the alternating group $A_4 = \langle (1\,2\,3), (1\,2)(3\,4) \rangle$.

(a) Find all conjugates of the cyclic subgroup $H = \langle (1\,2\,3) \rangle$, and state whether it is normal in A_4.

(b) Find all conjugates of the cyclic subgroup $K = \langle (1\,2)(3\,4) \rangle$, and state whether it is normal in A_4.

6. The subgroup lattice of D_4 is shown here:



For each of the 10 subgroups of D_4, find all of its conjugates, and determine whether it is normal in D_4. Fully justify your answers. [*Hint*: do this without computing xHx^{-1} for any subgroup H.]

Solution: Hint: The trivial group, the entire group, $\langle r^2\rangle$, and the three subgroups of order 4 are all normal, so the only conjugate of H from this list is H itself.

We only need to figure out the conjugate subgroups of the group of order 2 generated a reflection. The reflections f and $r^2 f$ are the same type, so they are in the same conjugacy class. Hence the two conjugates of $\langle f\rangle$ are itself and $\langle r^2 f\rangle$. Similarly, the two conjugates of $\langle r^2 f\rangle$ are itself and $\langle f\rangle$.

The two conjugates of $\langle rf\rangle$ are itself and $\langle r^3 f\rangle$. Similarly, the two conjugates of $\langle r^3 f\rangle$ are itself and $\langle rf\rangle$.

II: Consider a chain of subgroups $K \leq H \leq G$.

(a) Prove or disprove (with a counterexample): If $K \trianglelefteq G$, then $K \trianglelefteq H$.

Solution: Hint: This is true. Why?

(b) Prove or disprove (with a counterexample): If $K \trianglelefteq H \trianglelefteq G$, then $K \trianglelefteq G$.
Hint: Check whether this is true for D_4 whose subgroups are given above.

Solution: Hint: This is not true. Hint: For a simple example, choose K and H from the subgroups of D_4 given above. The edge between each arrow (the index of H in K) is 2, so each subgroup K is normal in subgroup H whenever there is an edge between them. However, the subgroup generated by just one reflection is not normal in D_4.

7. Let H be a subgroup of G. Given two fixed elements $a, b \in G$, define the sets

$$aHbH = \{ah_1bh_2 \mid h_1, h_2 \in H\} \qquad \text{and} \qquad abH = \{abh \mid h \in H\}.$$

Prove that if $H \trianglelefteq G$, then $aHbH = abH$.

8. Prove that $A \times \{e_B\}$ is a normal subgroup of $A \times B$, where e_B is the identity element of B.

9. All of the following statements are *false*. For each one, exhibit an explicit counterexample, and justify your reasoning. Assume that each $H_1 \trianglelefteq G_1$ and $H_2 \trianglelefteq G_2$.

   (a) If H and G/H is abelian, then G is abelian.

   Hint: A smallest counterexample would be to let G be a non-abelian group of order 6.

   (b) If every proper subgroup H of a group G is cyclic, then G is cyclic.

   Solution: Hint: Let G be a non-cyclic group of order 4 or 6.

   (c) If $G_1 \cong G_2$ and $H_1 \cong H_2$, then $G_1/H_1 \cong G_2/H_2$.

   Solution: A counterexample (where G_1 and G_2 are finite groups) is given earlier in this PDF file

   Solution: Another counterexample is to consider $G_1 = G_2 = \mathbb{Z}$ and its subgroups $H_1 = 4\mathbb{Z}$ and $H_2 = 5\mathbb{Z}$. Then H_1 and H_2 are isomorphic because they are both cyclic groups of infinite order. But $G_1/H_1 = \mathbb{Z}/4\mathbb{Z}$ has order 4 while $G_2/H_2 = \mathbb{Z}/5\mathbb{Z}$ has order 5.

   (d) If $H_1 \cong H_2$ and $G_1/H_1 \cong G_2/H_2$, then $G_1 \cong G_2$.

   Hint: A smallest counterexample is to take non-isomorphic groups G_1 and G_2 which are both of order 4.