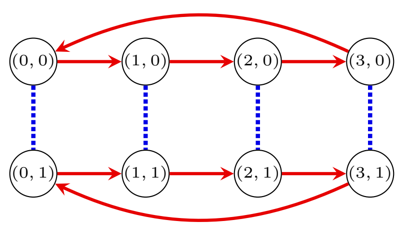
1. Prove or disprove: The groups $4\mathbb{Z}$ and $5\mathbb{Z}$ are isomorphic.

2. **Question:**

   If H and K are subgroups and $H \cong K$, then are G/H and G/K isomorphic?

   To answer this question, consider the group $G := \mathbb{Z}_4 \times \mathbb{Z}_2$.
   Below is the Cayley diagram for G with two generators,

   $(1, 0)$ (of order 4, solid arrow) and $(0, 1)$ (of order 2, dotted edge):



   We can visually demonstrate (using the above Cayley diagram) that the quotient of $\mathbb{Z}_4 \times \mathbb{Z}_2$ by the subgroup $H = \langle(0, 1)\rangle$ is the cyclic group $\mathbb{Z}_4$. To do this, collapse all the dotted edges representing H.

   (a) List all elements (cosets) in the quotient group $G/H = (\mathbb{Z}_4 \times \mathbb{Z}_2)/\langle\ (0, 1)\ \rangle$ (or circle them).

   (b) The quotient of $G = \mathbb{Z}_4 \times \mathbb{Z}_2$ by the subgroup $K = \langle(2, 0)\rangle$ is a harder to see. List all elements in the quotient group G/K.

   (c) What familiar group is $G/K = (\mathbb{Z}_4 \times \mathbb{Z}_2)/\langle(2, 0)\rangle$ isomorphic to?

3. The *center* of a group G is the set

   $$Z(G) = \{z \in G \mid gz = zg, \text{ for all } g \in G\} = \{z \in G \mid gzg^{-1} = z, \text{ for all } g \in G\}.$$

   We proved in an earlier homework that $Z(G)$ is a subgroup of G. Now, prove that $Z(G)$ is normal in G.

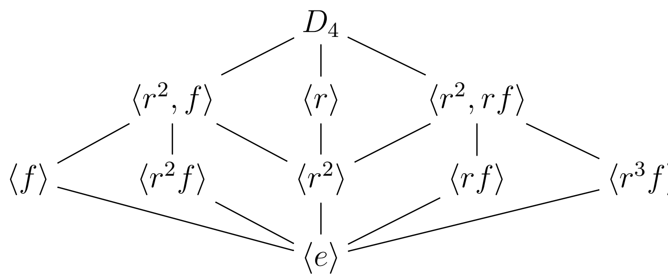
4. Suppose $f : G \to H$ is a group homomorphism. Prove that the subgroup $\ker f$ is normal in G.

5. Consider the alternating group $A_4 = \langle(1\,2\,3), (1\,2)(3\,4)\rangle$.

   (a) Find all conjugates of the subgroup $H = \langle(1\,2\,3)\rangle$, and state whether it is normal in A_4.

   (b) Find all conjugates of the subgroup $K = \langle(1\,2)(3\,4)\rangle$, and state whether it is normal in A_4.

6. The subgroup lattice of D_4 is shown here:



For each of the 10 subgroups of D_4, find all of its conjugates, and determine whether it is normal in D_4. Fully justify your answers. [*Hint*: do this without computing xHx^{-1} for any subgroup H.]

II: Consider a chain of subgroups $K \leq H \leq G$.

(a) Prove or disprove (with a counterexample): If $K \trianglelefteq G$, then $K \trianglelefteq H$.

(b) Prove or disprove (with a counterexample): If $K \trianglelefteq H \trianglelefteq G$, then $K \trianglelefteq G$.
Hint: Check whether this is true for D_4 whose subgroups are given above.

7. Let H be a subgroup of G. Given two fixed elements $a, b \in G$, define the sets

$$aHbH = \{ah_1bh_2 \mid h_1, h_2 \in H\} \qquad \text{and} \qquad abH = \{abh \mid h \in H\}.$$

Prove that if $H \trianglelefteq G$, then $aHbH = abH$.

8. Prove that $A \times \{e_B\}$ is a normal subgroup of $A \times B$, where e_B is the identity element of B.

9. All of the following statements are *false*. For each one, exhibit an explicit counterexample, and justify your reasoning. Assume that each $H_1 \trianglelefteq G_1$ and $H_2 \trianglelefteq G_2$.

(a) If H and G/H is abelian, then G is abelian.
Hint: A smallest counterexample would be to let G be a non-abelian group of order 6.

(b) If every proper subgroup H of a group G is cyclic, then G is cyclic.

(c) If $G_1 \cong G_2$ and $H_1 \cong H_2$, then $G_1/H_1 \cong G_2/H_2$.

(d) If $H_1 \cong H_2$ and $G_1/H_1 \cong G_2/H_2$, then $G_1 \cong G_2$.
Hint: A smallest counterexample is to take non-isomorphic groups G_1 and G_2 which are both of order 4.