1. Let H denote the subgroup $\{1, -1\}$ of the multiplicative group $\mathbb{R}^*$. Let G be a subgroup of S_n, and define a function $f : G \to H$ by

$$f(w) = \begin{cases} 1 & \text{if } w \text{ is an even permutation} \\ -1 & \text{if } w \text{ is an odd permutation} \end{cases}$$

(a) Prove that f is a homomorphism from G to H.

Solution:

$$f(vw) = \begin{cases} 1 & \text{if } w \text{ and } v \text{ have the same parity} \\ -1 & \text{if } w \text{ and } v \text{ have opposite parity} \end{cases},$$

and

$$f(v)f(w) = \begin{cases} 1 & \text{if } w \text{ and } v \text{ have the same parity} \\ -1 & \text{if } w \text{ and } v \text{ have opposite parity.} \end{cases}$$

(b) What is the kernel of f?

Solution: $\ker f$ is the set of even permutations in G

(c) If we let $G = \langle (13), (24) \rangle$ be the subgroup of S_5 generated by (13) and (24), what is $\ker f$ and $\text{Im} f$?

(d) If $G = \langle (13)(24), (12)(34) \rangle$ is the subgroup of S_5 generated by (13)(24) and (12)(34), what is $\ker f$ and $\text{Im} f$?

Solution: Since the group G is $\{Id, (13)(24), (12)(34), (14)(23)\}$, all permutations in G are even.
The kernel of f is the entire group G, and the image of f is the trivial subgroup $\{1\}$.

(e) What is the kernel of f and the image of f if we let $G = \langle (12)(345) \rangle$ be the subgroup of S_5 generated by (12)(345)?

2. Consider the map $\phi : \mathbb{C}^* \to \mathbb{C}^*$ defined by

$$\phi(z) = z^4$$

a.) Prove that ϕ is a homomorphism.

Solution: HW06

b.) List the elements in the kernel of ϕ.

Note: Since 1 is the identity element in $\mathbb{C}^*$, the kernel of ϕ is $\ker \phi = \{z \in \mathbb{C}^* : \phi(z) = 1\}$.

Solution: HW06

c.) Is ϕ an isomorphism? (If yes, prove that it is both surjective *and* injective; if no, prove that it's not injective *or* not surjective.)

Solution: HW06

3. Consider the map $\psi : \mathbb{Z}_{12} \to \mathbb{Z}_{10}$ defined by

$$\psi(m) = 3m$$

a.) Prove that ψ is *not* a homomorphism.

Solution: We have $\psi(6+6) = 3(0) = 0$, but $\psi(6) + \psi(6) = 8 + 8 = 6$. So $\psi(6+6) \neq \psi(6) + \psi(6)$.

4. Consider the map $\psi : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by

$$\psi(m) = 3m$$

a.) Prove that ψ is a homomorphism.

Solution: Suppose $a, b \in \mathbb{Z}_{12}$. Then $\psi(a+b) = 3(a+b) = 3a + 3b = \psi(a) + \psi(b)$.

b.) List the elements in the kernel of ψ.

Note: Since 0 is the identity element in $\mathbb{Z}_{12}$, the kernel of ψ is $\ker\psi = \{m \in \mathbb{Z}_{12} : \psi(m) = 0\}$.

Solution: The elements in $\ker\psi$ are the elements $a \in \mathbb{Z}_{12}$ such that $3a$ is congruent to 0 modulo 12, that is, $3a - 0$ is divisible by 12. So

$$\ker\psi = \{0, 4, 8\}$$

c.) Is ψ an isomorphism? (If yes, prove that it is both surjective *and* injective; if no, prove that it's not injective *or* prove that it is not surjective.)

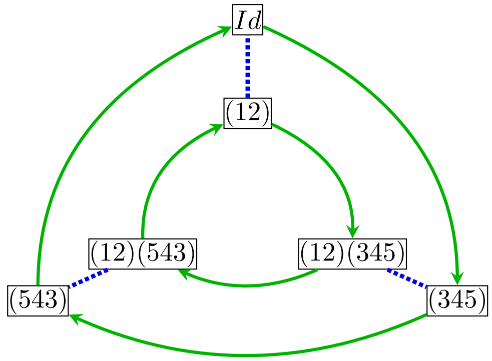
Solution: No, the function ψ is not an isomorphism. For example, we know ψ is not injective since $\psi(0) = 0 = \psi(4)$ although $0 \neq 4$ in $\mathbb{Z}_{12}$.

5. Let $H = \langle (12), (345) \rangle$ denote the subgroup of S_5 generated by (12) and (345).

Prove or disprove: There is an isomorphism from $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ to H.

Solution: True.
First, let's find all elements of $H = \langle (12), (345) \rangle$ by drawing its Cayley diagram using the generating set $\{(12), (345)\}$. By definition, H is the set of all products of $(12), (345)$, and their inverses.

Below is the Cayley graph for H with $S = \{(12), (345)\}$ as the generating set. Each solid (green) arrow has label (345). Each dotted (blue) edge has label (12).



We found that $H = \{Id, (12)(345), (543), (12), (345), (12)(543)\}$, which is equal to

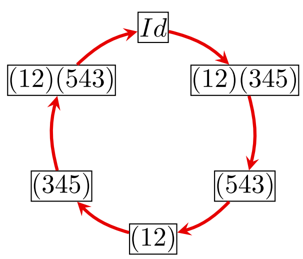
$$\{Id, c, c, c^2, c^3, c^4, c^5\}, \text{ where } c = (12)(345).$$

So H is a cyclic group of order 6. Every cyclic group of order 6 is isomorphic to $\mathbb{Z}_6$, so H is isomorphic to $\mathbb{Z}_6$ (meaning there exists an isomorphism between H and $\mathbb{Z}_6$).

We will now explicitly define an isomorphism from $\mathbb{Z}_6$ to H. Let $f : \mathbb{Z}_6 \to H$ be defined by

$$f(x) = c^x$$

Below is the Cayley graph for H with $c = (12)(345)$ as the generator (so here the generating set S is the singleton set $\{c\}$). Each solid (red) arrow is labeled by $c = (12)(345)$.



6. Prove or disprove: there is an isomorphism from $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ to $U(10) = \{1, 3, 7, 9\}$.

Solution: True.
Note that $\langle 3 \rangle = \{3^0, 3^1, 3^3, 3^2\} = \{1, 3, 7, 9\} = U(10)$, so $U(10)$ is a cyclic group which can be generated by 3. (Another possible generator is 7.) Every cyclic group of order 4 is isomorphic to $\mathbb{Z}_4$, so $U(10)$ is isomorphic to $\mathbb{Z}_4$ (meaning there exists an isomorphism between $U(10)$ and $\mathbb{Z}_4$).

We can explicitly define an isomorphism. Let $f : \mathbb{Z}_4 \to U(10)$ be defined by

$$f(x) = 3^x$$

7. Let J denote the subgroup of S_5 generated by (13) and (24). Prove or disprove: there is an isomorphism from $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ to $J = \langle (13), (24) \rangle$.

Solution: False.
First, let's find all elements of J. By definition, J is the set of all products of $(13), (24)$, and their inverses. We find that $J = \{Id, (13), (24), (13)(24)\}$. Observe that $\langle (13) \rangle = \{Id, (13)\}$, $\langle (24) \rangle = \{Id, (24)\}$, and $\langle (13)(24) \rangle = \{Id, (13)(24)\}$, so no one element of J can generate the entire group. Therefore, J is not a cyclic group.

Proof that there is no isomorphism from $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ to J:
Suppose there is an isomorphism $f : \mathbb{Z}_4 \to J$.

- Case $f(1) = Id$: Then $f(2) = f(1+1) = f(1)f(1) = Id \cdot Id = Id = f(1)$. Having $f(2) = f(1)$ means f is not injective.
- Case $f(1) \neq Id$: Then $f(1) = x$ where $x = (13)$, (24), or $(13)(24)$. Then

$$\begin{aligned} f(3) &= f(1+1+1) \\ &= f(1)f(1)f(1) \\ &= x^3 \\ &= x^2 x \\ &= x \text{ since we checked above that } x^2 = Id \\ &= f(1) \end{aligned}$$

  Having $f(3) = f(1)$ means f is not injective.

In both cases, f is not a bijection. So there is no isomorphism from $\mathbb{Z}_4$ to J.

8. Prove or disprove: there is an isomorphism from $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ to $U(8) = \{1, 3, 5, 7\}$.

Solution: HW06

9. Prove or disprove: there is an isomorphism from $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ to $\mathbb{T}_4$, where

$$\mathbb{T}_4 \text{ is the 4-th roots of unity } \{1, i, -1, -i\} = \{1, e^{i(\pi/2)}, e^{i(\pi/2)2}, e^{i(\pi/2)3}\}.$$

Solution: HW06

10. (a) Complete the table so that it depicts the Cayley table ("multiplication" table) of a group $G = \{e, x, y, z\}$, with e as the identity element.
There may be more than one way to complete a table, and if so you need to give all possibilities.
(Note: Many copies of this table are printed below so that you can use them for your scratch work.)

	e	x	y	z
e				
x				
y			e	
z				

	e	x	y	z
e				
x				
y			e	
z				

	e	x	y	z
e				
x				
y			e	
z				

	e	x	y	z
e				
x				
y			e	
z				

Solution: One is the Klein 4 group and the other is the cyclic group of order 4.

(b) Circle one of the tables you have completed. Write down a minimal generating set.

(c) Draw the Cayley diagram for the minimal generating set that you wrote above.

(d) What is the order of the element y in the group whose Cayley table you circled above?

11. Fill in the table so that it depicts the Cayley table of a group $G = \{e, a, b\}$, with e as the identity element. There may be more than one way to complete a table, and if so you need to give all possibilities.

(Note: Many copies of this table are printed below so that you can use them for your scratch work.)

	e	a	b
e			
a			
b			

	e	a	b
e			
a			
b			

	e	a	b
e			
a			
b			

	e	a	b
e			
a			
b			

(a) Circle one of the tables you have completed. Write down a minimal generating set.

(b) Draw the Cayley diagram for the minimal generating set that you wrote above.

(c) What is the order of the element b in the group whose Cayley table you circled above?

12. Let G and H be groups, and let e_G and e_H denote their identity elements. Let $f : G \to H$ be a homomorphism of groups.

(a) Prove that f sends e_G to e_H.

Solution: We have

$$\begin{aligned} e_H f(e_G) &= f(e_G) \text{ since } e_H \text{ is the identity element in } H \\ &= f(e_G\, e_G) \text{ since } e_G \text{ is the identity element in } G \\ &= f(e_G) f(e_G) \text{ since } f \text{ is a homomorphism} \end{aligned}$$

Multiplying on the left by $f(e_G)^{-1}$ of $e_H f(e_G) = f(e_G)f(e_G)$, we get $e_H = f(e_G)$.

(b) For each $x \in G$, prove that the inverse of $f(x)$ in H is $f(x^{-1})$.

Solution: See Judson textbook Proposition 11.4 for parts (a), (b), (c), (d)

(c) Let K be a subgroup of G. Prove that the image $f(K)$ is a subgroup of H.

(d) Let J be a subgroup of H. Prove that the preimage $f^{-1}(J)$ is a subgroup of G.

(e) Prove that $\ker f$ is a subgroup of G.

Solution: (See week 6 class notes and Judson textbook Theorem 11.5)