1. Let H denote the subgroup $\{1, -1\}$ of the multiplicative group $\mathbb{R}^*$. Let G be a subgroup of S_n, and define a function $f : G \to H$ by

$$f(w) = \begin{cases} 1 & \text{if } w \text{ is an even permutation} \\ -1 & \text{if } w \text{ is an odd permutation} \end{cases}$$

   (a) Prove that f is a homomorphism from G to H.
   (b) What is the kernel of f?
   (c) If we let $G = \langle (13), (24) \rangle$ be the subgroup of S_5 generated by (13) and (24), what is $\ker f$ and $\mathrm{Im} f$?
   (d) If $G = \langle (13)(24), (12)(34) \rangle$ is the subgroup of S_5 generated by (13)(24) and (12)(34), what is $\ker f$ and $\mathrm{Im} f$?
   (e) What is the kernel of f and the image of f if we let $G = \langle (12)(345) \rangle$ be the subgroup of S_5 generated by (12)(345)?

2. Consider the map $\phi : \mathbb{C}^* \to \mathbb{C}^*$ defined by

$$\phi(z) = z^4$$

   a.) Prove that ϕ is a homomorphism.
   b.) List the elements in the kernel of ϕ.

   Note: Since 1 is the identity element in $\mathbb{C}^*$, the kernel of ϕ is $\ker \phi = \{z \in \mathbb{C}^* : \phi(z) = 1\}$.

   c.) Is ϕ an isomorphism? (If yes, prove that it is both surjective *and* injective; if no, prove that it's not injective *or* not surjective.)

3. Consider the map $\psi : \mathbb{Z}_{12} \to \mathbb{Z}_{10}$ defined by

$$\psi(m) = 3m$$

   a.) Prove that ψ is *not* a homomorphism.

4. Consider the map $\psi : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by

$$\psi(m) = 3m$$

   a.) Prove that ψ is a homomorphism.
   b.) List the elements in the kernel of ψ.

   Note: Since 0 is the identity element in $\mathbb{Z}_{12}$, the kernel of ψ is $\ker \psi = \{m \in \mathbb{Z}_{12} : \psi(m) = 0\}$.

   c.) Is ψ an isomorphism? (If yes, prove that it is both surjective *and* injective; if no, prove that it's not injective *or* prove that it is not surjective.)

5. Let $H = \langle (12), (345) \rangle$ denote the subgroup of S_5 generated by (12) and (345).

   Prove or disprove: There is an isomorphism from $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ to H.

6. Prove or disprove: there is an isomorphism from $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ to $U(10) = \{1, 3, 7, 9\}$.

7. Let J denote the subgroup of S_5 generated by (13) and (24). Prove or disprove: there is an isomorphism from $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ to $J = \langle (13), (24) \rangle$.

8. Prove or disprove: there is an isomorphism from $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ to $U(8) = \{1, 3, 5, 7\}$.

9. Prove or disprove: there is an isomorphism from $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ to $\mathbb{T}_4$, where

$$\mathbb{T}_4 \text{ is the 4-th roots of unity } \{1, i, -1, -i\} = \{1, e^{i(\pi/2)}, e^{i(\pi/2)2}, e^{i(\pi/2)3}\}.$$

10. (a) Complete the table so that it depicts the Cayley table ("multiplication" table) of a group $G = \{e, x, y, z\}$, with e as the identity element.

There may be more than one way to complete a table, and if so you need to give all possibilities.

(Note: Many copies of this table are printed below so that you can use them for your scratch work.)

	e	x	y	z
e				
x				
y			e	
z				

	e	x	y	z
e				
x				
y			e	
z				

	e	x	y	z
e				
x				
y			e	
z				

	e	x	y	z
e				
x				
y			e	
z				

(b) Circle one of the tables you have completed. Write down a minimal generating set.

(c) Draw the Cayley diagram for the minimal generating set that you wrote above.

(d) What is the order of the element y in the group whose Cayley table you circled above?

11. Fill in the table so that it depicts the Cayley table of a group $G = \{e, a, b\}$, with e as the identity element. There may be more than one way to complete a table, and if so you need to give all possibilities.

(Note: Many copies of this table are printed below so that you can use them for your scratch work.)

	e	a	b
e			
a			
b			

	e	a	b
e			
a			
b			

	e	a	b
e			
a			
b			

	e	a	b
e			
a			
b			

(a) Circle one of the tables you have completed. Write down a minimal generating set.

(b) Draw the Cayley diagram for the minimal generating set that you wrote above.

(c) What is the order of the element b in the group whose Cayley table you circled above?

12. Let G and H be groups, and let e_G and e_H denote their identity elements. Let $f : G \to H$ be a homomorphism of groups.

(a) Prove that f sends e_G to e_H.

(b) For each $x \in G$, prove that the inverse of $f(x)$ in H is $f(x^{-1})$.

(c) Let K be a subgroup of G. Prove that the image $f(K)$ is a subgroup of H.

(d) Let J be a subgroup of H. Prove that the preimage $f^{-1}(J)$ is a subgroup of G.

(e) Prove that $\ker f$ is a subgroup of G.