1. Let H denote the subgroup  $\{1, -1\}$  of the multiplicative group  $\mathbb{R}^*$ . Let G be a subgroup of  $S_n$ , and define a function  $f: G \to H$  by

$$f(w) = \begin{cases} 1 & \text{if } w \text{ is an even permutation} \\ -1 & \text{if } w \text{ is an odd permutation} \end{cases}$$

- (a) Prove that f is a homomorphism from G to H.
- (b) What is the kernel of f?
- (c) If we let  $G = \langle (13), (24) \rangle$  be the subgroup of  $S_5$  generated by (13) and (24), what is ker f and Im f?
- (d) If  $G = \langle (13)(24), (12)(34) \rangle$  is the subgroup of  $S_5$  generated by (13)(24) and (12)(34), what is ker f and Im f?
- (e) What is the kernel of f and the image of f if we let  $G = \langle (12)(345) \rangle$  be the subgroup of  $S_5$  generated by (12)(345)?
- 2. Consider the map  $\phi : \mathbb{C}^* \to \mathbb{C}^*$  defined by

$$\phi(z) = z^4$$

- a.) Prove that  $\phi$  is a homomorphism.
- b.) List the elements in the kernel of  $\phi$ .

Note: Since 1 is the identity element in  $\mathbb{C}^*$ , the kernel of  $\phi$  is ker  $\phi = \{z \in \mathbb{C}^* : \phi(z) = 1\}$ .

c.) Is  $\phi$  an isomorphism? (If yes, prove that it is both surjective *and* injective; if no, prove that it's not injective *or* not surjective.)

3. Consider the map  $\psi : \mathbb{Z}_{12} \to \mathbb{Z}_{10}$  defined by

$$\psi(m) = 3m$$

- a.) Prove that  $\psi$  is *not* a homomorphism.
- 4. Consider the map  $\psi : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$  defined by

$$\psi(m) = 3m$$

- a.) Prove that  $\psi$  is a homomorphism.
- b.) List the elements in the kernel of  $\psi$ .

Note: Since 0 is the identity element in  $\mathbb{Z}_{12}$ , the kernel of  $\psi$  is ker  $\psi = \{m \in \mathbb{Z}_{12} : \psi(m) = 0\}$ .

c.) Is  $\psi$  an isomorphism? (If yes, prove that it is both surjective *and* injective; if no, prove that it's not injective *or* prove that it is not surjective.)

5. Let  $H = \langle (12), (345) \rangle$  denote the subgroup of  $S_5$  generated by (12) and (345).

Prove or disprove: There is an isomorphism from  $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$  to H.

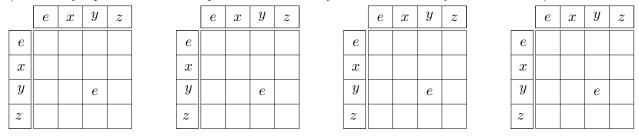
- 6. Prove or disprove: there is an isomorphism from  $\mathbb{Z}_4 = \{0, 1, 2, 3\}$  to  $U(10) = \{1, 3, 7, 9\}$ .
- 7. Let J denote the subgroup of  $S_5$  generated by (13) and (24). Prove or disprove: there is an isomorphism from  $\mathbb{Z}_4 = \{0, 1, 2, 3\}$  to  $J = \langle (13), (24) \rangle$ .

- 8. Prove or disprove: there is an isomorphism from  $\mathbb{Z}_4 = \{0, 1, 2, 3\}$  to  $U(8) = \{1, 3, 5, 7\}$ .
- 9. Prove or disprove: there is an isomorphism from  $\mathbb{Z}_4 = \{0, 1, 2, 3\}$  to  $\mathbb{T}_4$ , where

 $\mathbb{T}_4$  is the 4-th roots of unity  $\{1, i, -1, -i\} = \{1, e^{i(\pi/2)}, e^{i(\pi/2)2}, e^{i(\pi/2)3}\}.$ 

10. (a) Complete the table so that it depicts the Cayley table ("multiplication" table) of a group  $G = \{e, x, y, z\}$ , with e as the identity element.

There may be more than one way to complete a table, and if so you need to give all possibilities. (Note: Many copies of this table are printed below so that you can use them for your scratch work.)



- (b) Circle one of the tables you have completed. Write down a minimal generating set.
- (c) Draw the Cayley diagram for the minimal generating set that you wrote above.
- (d) What is the order of the element y in the group whose Cayley table you circled above?
- 11. Fill in the table so that it depicts the Cayley table of a group  $G = \{e, a, b\}$ , with e as the identity element. There may be more than one way to complete a table, and if so you need to give all possibilities.

	e	a	b		e	a	b		e	a	b		e	a	b
e				e				e				e			
a				a				a				a			
b				b				b				b			

(Note: Many copies of this table are printed below so that you can use them for your scratch work.)

- (a) Circle one of the tables you have completed. Write down a minimal generating set.
- (b) Draw the Cayley diagram for the minimal generating set that you wrote above.
- (c) What is the order of the element b in the group whose Cayley table you circled above?
- 12. Let G and H be groups, and let  $e_G$  and  $e_H$  denote their identity elements. Let  $f : G \to H$  be a homomorphism of groups.
  - (a) Prove that f sends  $e_G$  to  $e_H$ .
  - (b) For each  $x \in G$ , prove that the inverse of f(x) in H is  $f(x^{-1})$ .
  - (c) Let K be a subgroup of G. Prove that the image f(K) is a subgroup of H.
  - (d) Let J be a subgroup of H. Prove that the preimage  $f^{-1}(J)$  is a subgroup of G.
  - (e) Prove that ker f is a subgroup of G.