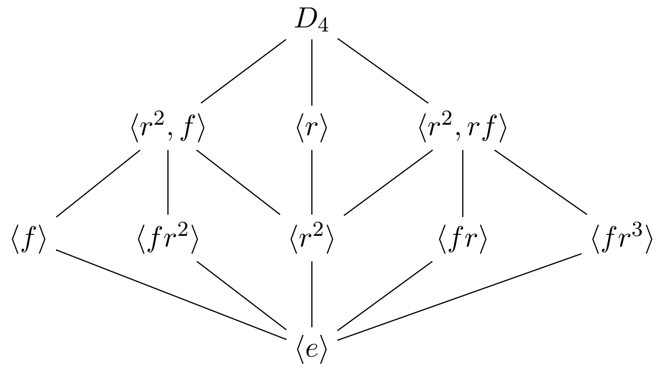
1. (a) The following is the "subgroup lattice" of D_4 Each edge between $K \leq H$ is labeled with the index, $[H : K]$. All labels are 2 (not drawn).



(b) List all elements in the subgroup $\langle r^2, f\rangle$.

> **Solution:** $\{e, r^2, f, r^2 f\}$
> (Note: $r^2 f$ is the same flip as fr^2 because of the relation $r^i f = fr^{-i}$)

(c) Is the left coset $r\langle r^2, f\rangle$ equal to the right coset $\langle r^2, f\rangle r$? Prove your answer.

> **Solution:** We know that the group D_4 has order 8. By Lagrange's theorem, since the order of $\langle r^2, f\rangle$ is 4, we get that $[D_4 : \langle r^2, f\rangle] = \frac{8}{4} = 2$. This implies that the left cosets of $\langle r^2, f\rangle$ and the right cosets of $\langle r^2, f\rangle$ coincide.
> (In general, $[G : H] = 2$ implies the left cosets and the right cosets are the same. The cosets are H and $G \setminus H$)

2. (a) Consider the subgroup $H = \{(1), (1,2)\}$ of S_3. Are all left cosets of H the same as all right cosets of H?

(b) Consider the subgroup $J = \{(1), (123), (132)\}$ of S_3. Are all left cosets of J the same as all right cosets of J?

(c) Let $n > 2$. Consider the subgroup A_n in S_n. Are all left cosets of A_n the same as the right cosets?

3. (a) What does it mean to say that a set S is a generating set of a group? What is a minimal generating set? Is it possible to have two different minimal generating sets for the same group? Give an example of a group with two different minimal generating sets.

(b) What do the vertices in a Cayley diagram represent?

> **Solution:** Each vertex represents an element in the group (so the number of vertices of the Cayley diagram is equal to the number of elements in the group).

(c) What do the arrows in a Cayley diagram represent?

> **Solution:** Each type of arrow (distinguished by color or label or texture) represents a generator.

4. What does it mean for a group to be abelian? What does it mean for a group to be non-abelian?

Solution: To show that a group is abelian, we must have $ab = ba$ for *all* elements a, b in the group.
To show that a group is *not* abelian, it is enough to find two elements c, d such that $cd \neq dc$.

How do you tell whether a group is abelian or non-abelian by looking at its Cayley graph?

Solution:

Note: To show that a group is abelian, it is enough to show that $xy = yx$ for all generators x, y. In particular, this means that if a group G can be generated by just one generator then G must be abelian.

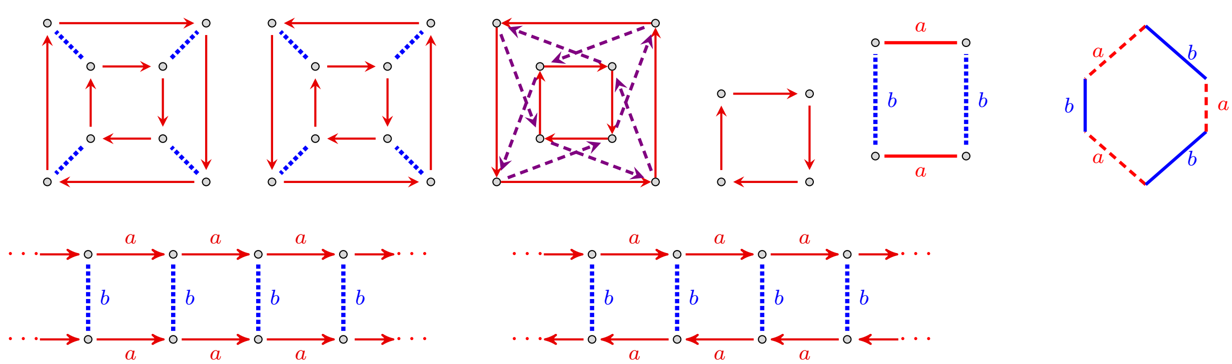
Note: The pattern on the left *never* appears in the Cayley graph for an abelian group, whereas the pattern on the right illustrates the relation $ab = ba$:

Note: You don't need to check more than one vertex of the Cayley graph. Every vertex has exactly the same patterns as all other vertices in the Cayley graph.

5. Below are Cayley diagrams of eight different groups (all have different group structures).

   (a) For each Cayley diagram, determine whether the corresponding group is abelian or non-abelian.

   (b) IF it is a Cayley diagram of a group shown in class, describe the group and the corresponding generators. (Six of the groups have been discussed during class so far.)



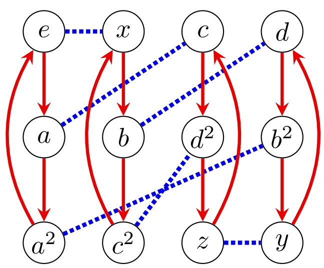
6. (a) Compute the orbit of the element r^2 in the group D_{10}, where r denotes the counterclockwise rotation by $\frac{2\pi}{10}$.

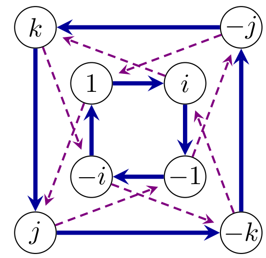
   Solution: $\{r^2, r^4, r^6, r^8, e\}$

   (b) Compute the orbit of the element 10 in the cyclic group $\mathbb{Z}_{16}$

   Solution: $\{10, 4, 14, 8, 2, 12, 6, 0\}$

7. Carry out the following steps for the groups whose Cayley graphs are shown below.





(a) Find the orbit of each element.

(b) Draw the orbit graph of the group.

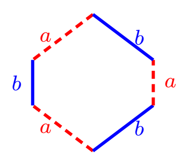
8. (a) Draw the Cayley graph for $\mathbb{Z}_6$ with $\{2, 3\}$ as a generating set.

   (b) Compare this with the Cayley graph for $\mathbb{Z}_3 \times \mathbb{Z}_2$ with $\{(1, 0), (0, 1)\}$ as a generating set. What can you conclude about the group $\mathbb{Z}_6$ and the group $\mathbb{Z}_3 \times \mathbb{Z}_2$?

   (c) Use the Cayley graph in class notes to help you find the subgroup of $\mathbb{Z}_3 \times \mathbb{Z}_2$ generated by the element $(1, 0)$. What is the subgroup generated by the element $(0, 1)$? What is the subgroup generated by the element $(2, 1)$?

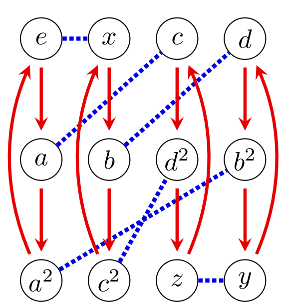
9. The group S_3 can be generated by the transpositions (1 2) and (2 3). Make a Cayley diagram for S_3 using this generating set.

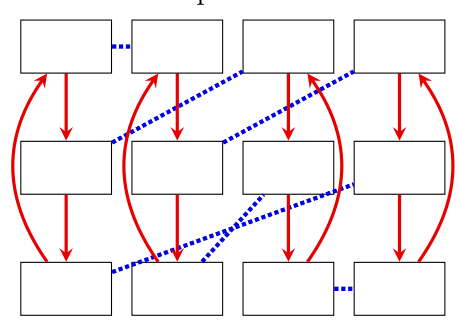
Solution: Let $a = (1\ 2)$ and $b = (2\ 3)$.



10. Make a Cayley diagram for the subgroup H of S_4 generated by the transpositions (1 2) and (3 4). What group is H the same as?

11. The following Cayley diagram for A_4 labels the elements with letters instead of permutations:





$A_4 = \{e, a, a^2, b, b^2, c, c^2, d, d^2, x, y, z\}$.

Redraw this Cayley diagram but label the nodes with the 12 even permutations in S_4. You need to determine which permutation corresponds to a, which to b, and so on.

Hint: There are many possible ways to do this. You should let a denote one of the permutations of order 3, and let x be an element of order 2 (for example, (12)(34)) that satisfies the pattern formed by the arrows, then determine the remaining elements.