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1 A cycle of odd length

- Warm-up: First compute $(15428)^2 = (15428)(15428)$.

> **Solution:** Warm-up: $(15428)^2 = (15428)(15428) = \boxed{(14852)}$

- Suppose σ is a cycle of odd length $(a_1\ a_2 \dots\ a_{2k}\ a_{2k+1})$. Compute σ^2.

> **Solution:** We now compute σ^2 in general. For $i = 1, 2, \dots, 2k-1$, we see that σ sends a_i to a_{i+1}, and σ sends a_{i+1} to a_{i+2}, so
>
> $$\sigma^2(a_i) = a_{i+2} \text{ for } i = 1, 2, \dots, 2k-1.$$
>
> We treat the other two indices $2k$ and $2k+1$ separately. We see that σ sends a_{2k} to a_{2k+1}, and σ sends a_{2k+1} to a_1, so
>
> $$\sigma^2(a_{2k}) = a_1.$$
>
> Similarly, σ sends a_{2k+1} to a_1, and σ sends a_1 to a_2, so
>
> $$\sigma^2(a_{2k+1}) = a_2.$$
>
> Therefore, $(a_1\ a_2 \dots\ a_{2k}\ a_{2k+1})^2 = \boxed{(a_1\ a_3\ a_5\ \dots a_{2k-1}\ a_{2k+1}\ a_2\ a_4\ a_6\ \dots\ a_{2k-2}\ a_{2k})}$.

- If σ is a cycle of odd length, prove that σ^2 is also a cycle.

> **Solution:** This is proven above.

2 Product of transpositions (lemma)

a. Warm-up: First try writing (25) as a finite product of $(12), (13), (14), \dots, (1n)$.
Write the transposition $(a\,b)$ as a finite product of $(12), (13), (14), \dots, (1n)$.

> **Solution:** Warm-up: $(25) = (12)(15)(12)$
>
> General solution:
> If $a = 1$ or $b = 1$, then $(a\,b)$ is already a product of $(12), (13), (14), \dots, (1n)$. Otherwise, we can write $(a\,b) = (1\,a)(1\,b)(1\,a)$

b. Warm-up: First try writing (25) as a finite product of $(12), (23), (34), \dots, (n-1, n)$.
Write the transposition $(a\,b)$ as a finite product of $(12), (23), (34), \dots, (n-1, n)$.

3 Exercise 26 of Judson Chapter 5

a. Prove that any permutation in S_n can be written as a finite product of $(12), (13), (14), \dots, (1n)$.

b. Prove that any permutation in S_n is a finite product of $(12), (23), (34), \dots, (n-1, n)$.

Solution: Every permutation in S_n can be written as the product of disjoint cycles (Textbook Theorem 5.9, stated in class but not proved). In class, we proved that every cycle is a product of transpositions: $(a_1\, a_2\, \dots\, a_k) = (a_1 a_2)(a_2 a_3)(a_3 a_4)\dots(a_{k-1} a_k)$. So every permutation can be written as a product of transpositions.

Since every transposition $(a\, b)$ in S_n can be written as a product of $(12), (13), \dots, (1n)$ by the previous exercise, every permutation in S_n can be written as a product of $(12), (13), \dots, (1n)$. Similarly, the previous exercise tells us that every transposition $(a\, b)$ in S_n can be written as a product of $(12), (23), (34), \dots, (n-1, n)$, every permutation in S_n can be written as a product of $(12), (23), (34), \dots, (n-1, n)$.

4 Conjugates

Let $\tau = (123\dots k)$.
a. Prove that if σ is any permutation, then $\sigma\tau\sigma^{-1} = (\sigma(1)\ \sigma(2)\ \sigma(3)\ \dots\ \sigma(k))$.
Hint: Note that $\sigma^{-1}(\sigma(i)) = i$. Now compute $\sigma\tau\sigma^{-1}(\sigma(i))$.

Solution: Suppose σ is a permutation in S_n. (We need to show that, for all $i = 1, 2, \dots, k-1$, the permutation $\sigma\tau\sigma^{-1}$ sends $\sigma(i)$ to $\sigma(i+1)$; we also need to show that $\sigma\tau\sigma^{-1}$ sends $\sigma(k)$ to $\sigma(1)$.)

If $i = 1, 2, \dots, k-1$, then we have

$$\begin{aligned} \sigma\tau\sigma^{-1}(\sigma(i)) &= \sigma\tau(i) \text{ since } \sigma^{-1}\sigma \text{ is the identity map} \\ &= \sigma(i+1) \text{ since } \tau \text{ sends } i \text{ to } i+1 \end{aligned}$$

We also have

$$\begin{aligned} \sigma\tau\sigma^{-1}(\sigma(k)) &= \sigma\tau(k) \text{ since } \sigma^{-1}\sigma \text{ is the identity map} \\ &= \sigma(1) \text{ since } \tau \text{ sends } k \text{ to } 1 \end{aligned}$$

This concludes the proof that $\sigma\tau\sigma^{-1} = (\sigma(1)\ \sigma(2)\ \sigma(3)\ \dots\ \sigma(k))$.

b. Let $\mu = (b_1\, b_2\, \dots\, b_k)$ be a cycle of length k. Prove that there is a permutation σ such that $\sigma\tau\sigma^{-1} = \mu$.

Solution: Since the previous part tells us that $\sigma\tau\sigma^{-1} = (\sigma(1)\ \sigma(2)\ \sigma(3)\ \dots\ \sigma(k))$, we can choose a permutation σ such that $\sigma(1) = b_1, \sigma(2) = b_2, \dots, \sigma(k) = b_k$.

5 Conjugation computation

Let $\tau = (1234)$.
(a) Let $\sigma = (135)(724)(89)$. Compute $\sigma\tau\sigma^{-1}$.

Solution: $\sigma\tau\sigma^{-1} = (135)(724)(89)\ (1234)\ (531)(427)(89) = (3457)$

(b) Let $\mu = (8275)$. Find a permutation σ such that $\sigma\tau\sigma^{-1} = \mu$.

Solution: We need to find a permutation such that

$$\begin{gathered} \sigma\tau\sigma^{-1} = \mu : \\ \sigma(1234)\sigma^{-1} = (8275) \end{gathered}$$

We can use the result $\sigma\tau\sigma^{-1} = (\sigma(1)\ \sigma(2)\ \sigma(3)\ \ldots\ \sigma(k))$ from the next exercise. Let σ be a permutation which sends 1 to 8, 2 to 2, 3 to 7, 4 to 5. For example, we can choose $\boxed{\sigma = (18)(37)(45)}$.

Then $\sigma\tau\sigma^{-1} = (18)(37)(45)(1234)(18)(37)(45) = (8275) = \mu$

6 Computation of cosets

(a) List the left and right cosets of the subgroup $3\mathbb{Z}$ in $\mathbb{Z}$.

Solution: The left and right cosets are the same since $\mathbb{Z}$ is abelian. They are $3\mathbb{Z}$, $1 + 3\mathbb{Z}$, $2 + 3\mathbb{Z}$.

(b) List the elements in the alternating group A_4.

Solution: The alternating group A_4 consists of the even permutations in S_4.

There are twelve even permutations in S_4. They are written down in the most recent class notes

(c) List the left and right cosets of the alternating subgroup A_4 in S_4.

Solution: Since every permutation is either even or odd, the left cosets are $A_4 = \{$the permutations from the previous exercise$\}$ and $(12)A_4 = \{$odd permutations$\}$. The right cosets are the same two sets.

7 Lemma 6.3

Prove the following **Lemma 6.3 in Judson Chapter 6:** Let H be a subgroup of a group G and suppose that $a, b \in G$. The following conditions are equivalent.

(1) $aH = bH$

(2) $Ha^{-1} = Hb^{-1}$

(3) $aH \subset bH$

(4) $b \in aH$

(5) $a^{-1}b \in H$

Solution: To prove that (3) implies (1), suppose $aH \subset bH$. So we have $a = a\,e = b\,h_1$ for some $h_1 \in H$, so

$$a\,h_1^{-1} = b. \tag{7.1}$$

Let $x \in bH$ (and we need to show that $x \in aH$). Then

$$\begin{aligned} x &= b\,h_2 \text{ for some } h_2 \in H \\ &= (a\,h_1^{-1})h_2 \text{ by (7.1)} \\ &= a(h_1^{-1}\,h_2) \in aH \text{ since } (h_1^{-1}\,h_2) \in aH \end{aligned}$$

Therefore $bH \subset aH$.

Solution: Proof for (1) implies (2) is in week 4 class notes. Proof for (4) implies (3) is in HW04.

Solution: To prove that (5) implies (4), suppose $a^{-1}b \in H$. Then $a^{-1}b = h$ for some $h \in H$. This means $b = ah \in aH$.

8 Conjugates and cosets

If $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$, prove that $gH = Hg$ for all $g \in G$ (that is, prove that the left cosets are identical to the right cosets). (HW04)

9 Index

(a) Prove the set $H = \{\text{Id}, (12)(34), (13)(24), (14)(23)\}$ is a subgroup of the symmetric group S_4.
(b) What is its index in S_4?

Solution: (a) Prove that the identity element is in H, that H is closed under the group operation, and H is closed under taking inverses. (b) By Lagrange's theorem, the index $[S_4 : H]$ is $\frac{|S_4|}{|H|} = \dfrac{4!}{4} = 3! = \boxed{6}$.

10 Lagrange's Theorem

(a) Suppose G is a finite group with an element g of order 3 and an element h of order 5. Why must $|G| \geq 15$?

Solution: The order of g and the order of h must both divide the order of G.

(b) Suppose G is a group of order 23. Describe G. Explain your answer.

Solution: (See the proof of Judson's Corollary 6.12)
Suppose $g \in G$ such that $g \neq e$. Since $\langle g \rangle$ is a subgroup of G, the number of elements in $\langle g \rangle$ must divide the number of elements in G (which is 23). Since $\langle g \rangle$ has more than one element (it contains at least g and e), we know that $\langle g \rangle$ is equal to 23.
Therefore, $G = \langle g \rangle$. So G is a cyclic group of order 23 (the same as $\mathbb{Z}_{23}$).