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1 A cycle of odd length

- Warm-up: First compute $(15428)^2 = (15428)(15428)$.
- Suppose σ is a cycle of odd length $(a_1\ a_2 \ldots\ a_{2k}\ a_{2k+1})$. Compute σ^2.
- If σ is a cycle of odd length, prove that σ^2 is also a cycle.

2 Product of transpositions (lemma)

a. Warm-up: First try writing (25) as a finite product of $(12), (13), (14), \ldots, (1n)$.
Write the transposition $(a\,b)$ as a finite product of $(12), (13), (14), \ldots, (1n)$.

b. Warm-up: First try writing (25) as a finite product of $(12), (23), (34), \ldots, (n-1, n)$.
Write the transposition $(a\,b)$ as a finite product of $(12), (23), (34), \ldots, (n-1, n)$.

3 Exercise 26 of Judson Chapter 5

a. Prove that any permutation in S_n can be written as a finite product of $(12), (13), (14), \ldots, (1n)$.

b. Prove that any permutation in S_n is a finite product of $(12), (23), (34), \ldots, (n-1, n)$.

4 Conjugates

Let $\tau = (123 \ldots k)$.
a. Prove that if σ is any permutation, then $\sigma\tau\sigma^{-1} = (\sigma(1)\ \sigma(2)\ \sigma(3)\ \ldots\ \sigma(k))$.
Hint: Note that $\sigma^{-1}(\sigma(i)) = i$. Now compute $\sigma\tau\sigma^{-1}(\sigma(i))$.
b. Let $\mu = (b_1\, b_2\, \ldots\, b_k)$ be a cycle of length k. Prove that there is a permutation σ such that $\sigma\tau\sigma^{-1} = \mu$.

5 Conjugation computation

Let $\tau = (1234)$.
(a) Let $\sigma = (135)(724)(89)$. Compute $\sigma\tau\sigma^{-1}$.
(b) Let $\mu = (8275)$. Find a permutation σ such that $\sigma\tau\sigma^{-1} = \mu$.

6 Computation of cosets

(a) List the left and right cosets of the subgroup $3\mathbb{Z}$ in $\mathbb{Z}$.

(b) List the elements in the alternating group A_4.

(c) List the left and right cosets of the alternating subgroup A_4 in S_4.

7 Lemma 6.3

Prove the following **Lemma 6.3 in Judson Chapter 6:** Let H be a subgroup of a group G and suppose that $a, b \in G$. The following conditions are equivalent.

(1) $aH = bH$

(2) $Ha^{-1} = Hb^{-1}$

(3) $aH \subset bH$

(4) $b \in aH$

(5) $a^{-1}b \in H$

8 Conjugates and cosets

If $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$, prove that $gH = Hg$ for all $g \in G$ (that is, prove that the left cosets are identical to the right cosets). (HW04)

9 Index

(a) Prove the set $H = \{\text{Id}, (12)(34), (13)(24), (14)(23)\}$ is a subgroup of the symmetric group S_4.
(b) What is its index in S_4?

10 Lagrange's Theorem

(a) Suppose G is a finite group with an element g of order 3 and an element h of order 5. Why must $|G| \geq 15$?
(b) Suppose G is a group of order 23. Describe G. Explain your answer.