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1 Computation

1.1 Arithmetic

Suppose a, b, c are elements of a group G, and suppose $|a| = 6$ and $|b| = 7$. Express

$$(a^4c^{-2}b^4)^{-1}a^3$$

without using negative exponents.

Solution:

$$\begin{aligned}(a^4c^{-2}b^4)^{-1}a^3 &= (b^4)^{-1}(c^{-2})^{-1}(a^4)^{-1}a^3 \text{ by the "socks-shoes" property} \\ &= b^3(c^{-2})^{-1}(a^2)a^3 \text{ since } b^4b^3 = b^7 = e \text{ and } a^4a^2 = a^6 = e \\ &= b^3c^2a^2a^3 \text{ after applying law of exponent } ((c^s)^r = c^{sr} \text{ for all } s, r \in \mathbb{Z}) \\ &= \boxed{b^3c^2a^5} \text{ after applying law of exponent } (a^sa^r = a^{s+r} \text{ for all } s, r \in \mathbb{Z})\end{aligned}$$

1.2 If a group element x is such that $x^{24} = e$, what are the possible orders of x?

Solution: 1,2,3,4,6,8,12,24 (all the divisors of 24)

1.3 Draw the subgroup lattice of $\mathbb{Z}_{24}$.

Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, and $\sigma = (126)(45) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 6 & 3 & 5 & 4 & 1 & 7 \end{bmatrix} \in S_7$

1.4 For each group below, if the group is finite, list all elements; if the group is infinite, describe all elements

1. $\langle$2D rotation by $240^o\rangle$ 2. $\langle A\rangle$ 3. $\langle B\rangle$ 4. $\langle AB\rangle$
5. $\langle\sigma\rangle$ 6. The symmetry group of a (non-square) rectangle, aka the "mattress group" from the first day

1.5 Match the groups from Question 1.4 with the groups below

(i) $\mathbb{Z}_2$ ______ (ii) $\mathbb{Z}_3$ ______ (iii) $\mathbb{Z}_4$ ______ (iv) $\mathbb{Z}_6$ ______ (v) $\mathbb{Z}$ ______ (vi) $\mathbb{Z}_2 \times \mathbb{Z}_2$ ______

Solution:
(i) $\langle B\rangle$ is the same as $\mathbb{Z}_2$ (ii) $\langle$2D rotation by $240^o\rangle$ is the same as $\mathbb{Z}_3$ (iii) $\langle A\rangle$ is the same as $\mathbb{Z}_4$ (iv) $\langle\sigma\rangle$ is the same as $\mathbb{Z}_6$ (v) $\langle AB\rangle$ is the same as $\mathbb{Z}$ (vi) The symmetry group of a (non-square) rectangle is the same as $\mathbb{Z}_2 \times \mathbb{Z}_2$

2 True or false

Let G denote a finite group. If the statement is true, give a proof. If it is false, give a counterexample.

2.1 For all elements $a, b \in G$, we have $|ab| = |ba|$

Solution: True. For a proof, see the sample proof in HW03.

2.2 For all elements $a, b \in G$, we have $|a| = |b^{-1}ab|$

Solution: True. A proof is part of your HW03.

2.3 For all elements $a, b \in G$, we have $|b| = |b^{-1}ab|$

Solution: False. For a counterexample, let $G = \langle x \rangle = \{e, x\}$ be the cyclic group of order two. The order of the identity element is 1 and the order of x is 2. Let $a = x$ and let $b = e$. Then $|b| = |e| = 1$ and $|b^{-1}ab| = |e^{-1}xe| = |x| = 2$.

3 Union of subgroups

3.1 Prove that it is impossible for a group G to be the union of two proper subgroups.

(In your proof, don't assume that the group is finite)

Solution: A possible proof: For the sake of contradiction, suppose $G = A \cup B$, where A and B are both proper subgroups of G.

Since B is proper, there is $a \in G$ such that $a \notin B$. Since $G = A \cup B$, it must be that $a \in A$. Similarly, there is $b \in B$ such that $b \notin A$.

Since $a, b \in G$, the closure property tells us that $ab \in G$. Thus ab is in A or in B (since $G = A \cup B$). If $ab \in A$, we have $b = a^{-1}(ab) \in A$, which is a contradiction. So $ab \in B$. But this means $a = (ab)b^{-1} \in B$, which is again a contradiction.

3.2 Let G denote the rectangle (non-square) "mattress group" (the symmetry group of a non-square rectangle). Write G as the union of three proper subgroups.

Solution: Let h and v denote the horizontal and vertical flips, and let r denote the rotation by 180^o. Then $\{Id, h\}$, $\{Id, v\}$, and $\{Id, r\}$ are all subgroups of G, and we have $G = \{Id, h\} \cup \{Id, v\} \cup \{Id, r\}$.

4 Subgroup

Find all subgroups of the symmetry group of the regular triangle (the "triangle mattress" group).