MATH 4210/5210 ALGEBRA PRACTICE WEEK 02

DO NOT SUBMIT

1. Definitions (Textbook Section 3.2)

(1) Given sets A, B, write the definition of the Cartesian product $A \times B$ (See Textbook Section 1.2)
(2) Write the definition of a binary operation on a set A
(3) Write the definition of an associative binary operation
(4) Write the definition of an identity element for a binary operation
(5) Write the definition of a group
(6) Write the definition of an inverse of an element a in a group $(G, \star)$

2. Textbook Proposition 3.21

Let G be a group, and let $a, b \in G$.

(1) Prove that the equation $ax = b$ has a solution in G and the solution is unique
(2) Prove that the equation $xa = b$ has a solution in G and the solution is unique

Proof. (1) (Exercise for student, or see Textbook's Proof of Prop. 3.21)

(2) First, we show that such an x exists. We see that a^{-1}, the inverse of a, exists in G (by property of a group). We also see that $ba^{-1} \in G$ by the closure property. Set $x = ba^{-1}$. Then

$$xa = (ba^{-1})a = b(a^{-1}a) = be = b,$$

so x is a solution in G.

To show that this solution is unique, suppose that x_1 and x_2 are both solutions of $xa = b$. Then $x_1 a = b = x_2 a$. So we have

$$\begin{aligned} x_1 &= x_1(aa^{-1}) \\ &= (x_1 a)a^{-1} \\ &= ba^{-1} \\ &= (x_2 a)a^{-1} \\ &= x_2(aa^{-1}) \\ &= x_2 \end{aligned}$$

This concludes the proof that the solution is unique.

□

3. Abelian and non-abelian groups

3.1. Square mattress group.

Consider the "Mattress Group" (see week 1 class notes). Since the mattress is in a non-square rectangle bed frame, there are exactly four transformations that can be done to the mattress (the identity, the 180^o rotation, the horizontal flip, and the vertical flip). Observe that $t_1 t_2 = t_2 t_1$ for all transformations t_1, t_2.

Now, suppose your mattress is a square mattress in a square bed frame.

(1) List all the transformations that can be done to this square mattress.
(2) Give an example of two transformations t_1, t_2 such that $t_1 t_2 \neq t_2 t_1$. (This means the "Square mattress" group is not abelian).

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3.2. **Group of bijections (from Group Quiz 01).** Consider the set G of all bijections $\{1,2,3\} \to \{1,2,3\}$. Then G is a group under function composition $\circ$.

Is $(G, \circ)$ abelian? (If so, prove it. Otherwise, show that the group operation is not commutative by finding two elements $a, b \in G$ such that $ab \neq ba$.)

3.3. **Section 3.5 Exercise 31.** Suppose G is a group and $a^2 = e$ for all elements $a \in G$. Prove that G is abelian.

Proof. (See Appendix B Hints and Answers to Selected Exercises) □

4. Right and left cancellation

Let G be a group, and let $a, b, c \in G$.

(1) (Right cancellation) Prove that $ba = ca$ implies $b = c$.
(2) (Left cancellation) Prove that $ab = ac$ implies $b = c$.

Proof. (1) Suppose $ba = ca$. Multiply on the right by a^{-1}:

$$(ba)a^{-1} = (ca)a^{-1}$$

Associativity give us

$$b(aa^{-1}) = a(a^{-1})$$
$$be = ce$$
$$b = c$$

(2) (Exercise for student)

□

5. Section 3.5 Exercise 7

Let $S = \mathbb{R}\backslash\{-1\}$. Define a binary operation on S by $a \star b = a + b + ab$. Show that $(S, \star)$ is an abelian group.

(1) Prove that this is indeed a binary operation on S, that is, show that if $a, b \in S$ then $a \star b \in S$
(2) Prove that $\star$ is associative
(3) Prove that S contains the identity of $\star$
(4) Prove that every element in S has an inverse under $\star$

Proof. (1) For the sake of contradiction, suppose that S is not closed under $\star$. So there exist $a, b \in S$ such that $a \star b \notin S$, that is, $a \star b = -1$. Then

$$-1 = a + b + ab$$

So we have

$$\begin{aligned} 0 &= 1 + a + b + ab \\ &= (1 + a) + b(1 + a) \\ &= (1 + b)(1 + a) \end{aligned}$$

Therefore, either $1 + b = 0$ or $1 + a = 0$. This implies $b = -1$ or $a = -1$, which contradicts the fact that $a, b \in S$. Hence S is closed under $\star$.

(2) (Prove that $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in S$)
(3) (What is the identity e for $\star$? Prove that $e \star a = a = a \star e$ for all $a \in S$.)
(4) (Give a formula for the inverse of each $a \in S$ under $\star$, and prove that this element is also in S.)

□

6. Odd integers (from group quiz 01)

Prove that the set of odd integers under addition is not a group.

Proof. The identity of addition is 0, which is not in the set of odd integers. (This fails the property of having an identity element.) □

Another possible proof. The set of odd integers is not closed under addition. For example, 1 and 3 are both in the set, but $1 + 3 = 4$ is not. (This fails the closure property.) □