Last updated: Oct 18, 2024

Abstract Algebra Notes Week 7 Wed, Oct 16 2024

External & internal direct products (Sec 9.2)

Given any groups A, B, we can construct the direct product $A \times B$.

This is called the external direct product (because the new group $A \times B$ is "outside" of either A and B)

We want to be able to reverse this process and produce an "internal" direct product, when possible.

Q: Given a group G, when can we write it as $G \cong H \times K$, where H and K are subgroups of G?

Ex 1 $G = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$

$H = \{0, 3\} = \langle 3 \rangle \cong \mathbb{Z}_2$

$K = \{0, 2, 4\} = \langle 2 \rangle \cong \mathbb{Z}_3$

The map $f: \mathbb{Z}_6 \longrightarrow \langle 3 \rangle \times \langle 2 \rangle$

$$\begin{aligned}
0 &\longmapsto (0, 0) \\
1 &\longmapsto (3, 4) \\
2 &\longmapsto (0, 2) \\
3 &\longmapsto (3, 0) \\
4 &\longmapsto (0, 4) \\
5 &\longmapsto (3, 2)
\end{aligned}$$

is an isomorphism.

Note:

$0 + 0 = 0$

$3 + 4 = 1$

$0 + 2 = 2$

$3 + 0 = 3$

$0 + 4 = 4$

$3 + 2 = 5$

all of G

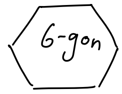
In this example,

$H + K \stackrel{\text{def}}{=} \{h + k : h \in H, k \in K\}$

is equal to G.

Ex 2 $G = D_6 = \{ Id, R, R^2, R^3, R^4, R^5, f, fR, fR^2, fR^3, fR^4, fR^5 \} = \langle f, R \rangle$

$R = Rot\left(\frac{2\pi}{6}\right) = Rot(60^\circ)$



$H = \{ Id, R^2, R^4, f, fR^2, fR^4 \} = \langle f, R^2 \rangle$

$K = \{ Id, R^3 \} = \langle R^3 \rangle$

The map $f : D_6 \longrightarrow \langle f, R^2 \rangle \times \langle R^3 \rangle$

$$\begin{array}{lcl}
Id & \mapsto & (Id, Id) \\
R & \mapsto & (R^4, R^3) \\
R^2 & \mapsto & (R^2, Id) \\
R^3 & \mapsto & (Id, R^3) \\
R^4 & \mapsto & (R^4, Id) \\
R^5 & \mapsto & (R^2, R^3) \\
f & \mapsto & (f, Id) \\
fR & \mapsto & (fR^4, R^3) \\
fR^2 & \mapsto & (fR^2, Id) \\
fR^3 & \mapsto & (f, R^3) \\
fR^4 & \mapsto & (fR^4, Id) \\
fR^5 & \mapsto & (fR^2, R^3)
\end{array}$$

is an isomorphism.

Note:

$$\begin{array}{llcl}
Id & Id & = & Id \\
R^4 & R^3 & = & R^7 = R \\
R^2 & Id & = & R^2 \\
Id & R^3 & = & R^3 \\
R^4 & Id & = & R^4 \\
R^2 & R^3 & = & R^5 \\
f & Id & = & f \\
fR^4 & R^3 & = & fR \\
fR^2 & Id & = & fR^2 \\
f & R^3 & = & fR^3 \\
fR^4 & Id & = & fR^4 \\
fR^2 & R^3 & = & fR^5
\end{array}$$

all of G

In this example,

$HK \overset{def}{=} \{ hk : h \in H, k \in K \}$

is equal to G

Conjugates of an element of a group

Def We say $x, y \in G$ are conjugate in G if $gxg^{-1} = y$ for some $g \in G$.

The element gxg^{-1} is called a conjugate of x

Prop • Two conjugate elts have the same order (HW 03)

• Conjugacy is an equivalence relation on G.

Conjugacy class of x is $\{gxg^{-1} : g \in G\}$

Conjugates of a subgroup of a group

Prop If $H \leq G$ and $g \in G$, the set

(Week 03 Practice Problems)

$$\underbrace{gHg^{-1}}_{\text{called a conjugate of } H} \overset{\text{def}}{=} \{ghg^{-1} : h \in H\}$$

is a subgroup.

Def • We say two subgroups H, K of G are conjugate in G

if $gHg^{-1} = K$ for some $g \in G$.

equality is required, not just isomorphism

Prop This relation is an equivalence relation on

the set of subgroups of G.

Conjugates of a subgroup of a group

Recall: In D_n, we have $R^n = Id$ and $fR^if = R^{-i}$
$(fR^i = R^{-i}f)$

Ex In D_4,

- $f\langle R\rangle f^{-1} = f\{Id, R, R^2, R^3\}f$

$$= \{f\,Id\,f,\ fRf,\ fR^2f,\ fR^3f\}$$

$$= \{Id,\ R^3,\ R^2,\ R\}$$

$$= \langle R\rangle$$

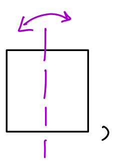
- $R\langle f\rangle R^{-1} = R\{Id, f\}R^{-1}$

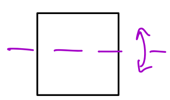
$$= \{R\,Id\,R^{-1},\ RfR^{-1}\}$$

$$= \{Id,\ fR^{-1}R^{-1}\}$$

$$= \{Id,\ fR^2\}$$

$$= \langle fR^2\rangle$$

If f is the flip ,

then fR^2 is the flip 

Normal subgroups (Sec 10.1)

<u>Def</u> Let G be a group and $H \leq G$ a subgroup.

We say H is <u>normal</u> if

$$gH = Hg$$

for all $g \in G$

(i.e. if all left and right cosets are the same)

Notation: $H \triangleleft G$ or $H \trianglelefteq G$

<u>Rem</u> If G is abelian, then any subgroup is normal.

<u>Ex</u> If $H \leq G$ with $[G:H] = 2$, then $H \trianglelefteq G$. So ...

* A_n is normal in S_n (half of S_n are even; half are odd)
* $\langle R \rangle = \underbrace{\{Id, R, R^2, \ldots, R^{n-1}\}}_{\text{half the elements in } D_n}$ is normal in D_n

<u>Ex</u> From HW 04 & 05, $H = \langle (ab) \rangle = \{Id, (ab)\}$ is $\underline{\underline{\text{not}}}$ normal in S_n for $n \geq 3$.

$(24)\langle(27)\rangle = \{(24), (274)\}$ but $\langle(27)\rangle(24) = \{(24), (247)\}$

Lemma The following are equivalent (TFAE):

1. $gH = Hg$ for all $g \in G$ (i.e. $H \trianglelefteq G$)

   "all left cosets are right cosets"

2. $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$

   "H is closed under conjugation"

3. $gHg^{-1} = H$ for all $g \in G$

   "H has only one conjugate subgroup, itself"

HW 04

Ex By the same reasoning as the above example for D_4, the subgroup $H = \langle f \rangle = \{Id, f\}$ of D_n $(n \geq 3)$ doesn't satisfy condition 3. For example,

$$R\langle f \rangle R^{-1} = \{Id, RfR^{-1}\}$$

$$\neq \langle f \rangle \quad \text{because} \quad RfR^{-1} = fR^{-1}R^{-1} = fR^{-2} = fR^{n-2} \neq f$$

So, by the lemma, $\langle f \rangle =$ is not normal in D_n.

$$\boxed{\begin{array}{l}\text{Recall Lemma for cosets:} \\ \quad a \in bH \text{ iff } aH = bH\end{array}}$$

(Lemma 6.3)

Q: Why normal subgroups?

Notation: If H is a subgroup of G, then

$$G/H \overset{\text{def}}{=} \{ gH : g \in G \}$$

is the set of left cosets of H in G.

A: It allows us the notion of internal direct product

A: We would like G/H to be a group with binary operation

$$(xH)(yH) = (xy)H.$$

Warning: In general, this is not well-defined.

Ex $G = D_4$, $H = \langle f \rangle$

We want

$$(RH)(R^3H) = R^4H = eH = H$$

since $R^4 = e$ in D_4

But $RH = RfH$ since

$$RH = R\{e, f\} = \{R, Rf\}$$

$$RfH = Rf\{e, f\} = \{Rf, R\}$$

So

$$\begin{aligned}(RH)(R^3H) &= (RfH)(R^3H) \\ &= RfR^3H \\ &= fR^{-1}R^3H \\ &= fR^2H\end{aligned}$$

But $fR^2H = fR^2\{e, f\} = \{fR^2, R^2\}$

So $fR^2H \neq H$.

In above example, the product $(xH)(yH)$ depends on the choice of coset representatives (x and y).

But normality fixes this issue.

Thm Let G be a group and $N \trianglelefteq G$.

(Thm 10.4) ① Coset multiplication in $G/N = \{\text{cosets of } N \text{ in } G\}$

$$(xN)(yN) = (xy)N$$

is a well-defined binary operation

(that is, the definition of coset multiplication depends on only the cosets and not on the coset representatives.)

② G/N is a group under the binary operation given above,

Read "G mod N"

called the quotient group (or the factor group) of G by N.

Proof ① Let $xN, yN \in G/N$ be cosets.

Suppose $xN = aN$ and $yN = bN$.

(We need to show $(xN)(yN) = xyN = abN = (aN)(bN)$)

Then $a \in xN$ and $b \in yN$.

So $a = xn_1$ and $b = yn_2$ for some $n_1, n_2 \in N$.

Hence $ab = x n_1 y n_2$

$= x y n_3 n_2$ for some $n_3 \in N$, since $n_1 y \in Ny = yN$ due to N being normal

$\in xyN$

Therefore $abN = xyN$ $\square$

② Part 1 tells us the binary operation is well-defined.

(Associativity) $((xN)(yN))(zN) = (xyN)(zN) = xyzN$

$(xN)((yN)(zN)) = (xN)(yzN) = xyzN$

(Identity) $eN = N$ is the identity

(Inverse) $x^{-1}N$ is the inverse of xN,

since $(xN)(x^{-1}N) = xx^{-1}N = eN = N$.

The quotient of $\mathbb{Z}$ by $4\mathbb{Z}$

Ex Let $G = \mathbb{Z}$, and $N := 4\mathbb{Z} = \{4k : k \in \mathbb{Z}\}$. Then $N \triangleleft G$.

The quotient group of $\mathbb{Z}$ by $4\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, consists of the following four cosets:

$$0 + 4\mathbb{Z} = 4\mathbb{Z} = \{\ldots, -4, 0, 4, 8, \ldots\}$$ the identity in G/N

$$1 + 4\mathbb{Z} = \{1 + 4k : k \in \mathbb{Z}\} = \{\ldots, -3, 1, 5, 9, \ldots\}$$

$$2 + 4\mathbb{Z} = \{2 + 4k : k \in \mathbb{Z}\} = \{\ldots, -2, 2, 6, 10, \ldots\}$$

$$3 + 4\mathbb{Z} = \{3 + 4k : k \in \mathbb{Z}\} = \{\ldots, -1, 3, 7, 11, \ldots\}$$

Note that $1+4\mathbb{Z}$ has order 4:

$$(1+4\mathbb{Z}) + (1+4\mathbb{Z}) = 2 + 4\mathbb{Z} \neq 4\mathbb{Z}$$

(Note: No elt can have order 3 because of Lagrange's Thm)

$$(1+4\mathbb{Z}) + (1+4\mathbb{Z}) + (1+4\mathbb{Z}) + (1+4\mathbb{Z}) = 4 + 4\mathbb{Z} = 4\mathbb{Z}$$

So $\langle 1+4\mathbb{Z} \rangle = \mathbb{Z}/4\mathbb{Z}$, and thus $\mathbb{Z}/4\mathbb{Z}$ is a cyclic group of order 4.

Therefore $\mathbb{Z}/4\mathbb{Z}$ is isomorphic to $\mathbb{Z}_4$.

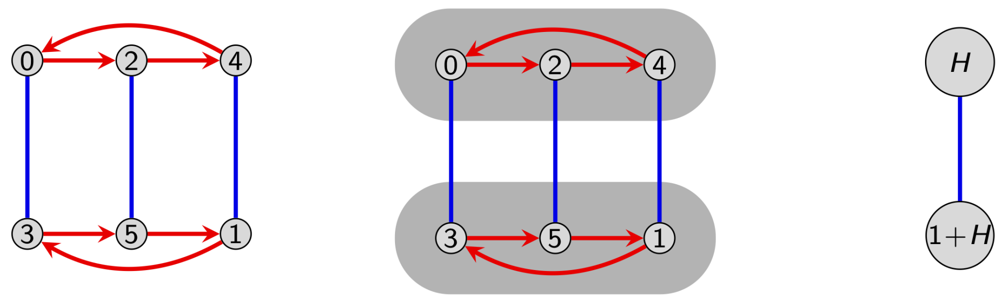
The quotient of $\mathbb{Z}_6$ by $\{0, 2, 4\}$

Ex

Consider the group $G = \mathbb{Z}_6$ and its normal subgroup $H = \langle 2 \rangle = \{0, 2, 4\} \cong \mathbb{Z}_3$

There are two (left) cosets: $H = \{0, 2, 4\}$ and $1 + H = \{1, 3, 5\}$.

The following diagram shows how to take a quotient of $\mathbb{Z}_6$ by H.



$\mathbb{Z}_6$ organized by the subgroup $H = \langle 2 \rangle$

Left cosets of H are near each other

Collapse cosets into single nodes

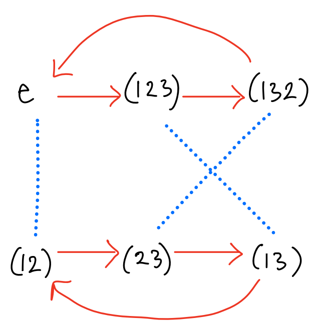
In this example, the resulting diagram *is* a Cayley diagram. So, we *can* "divide" $\mathbb{Z}_6$ by $\langle 2 \rangle$, and we see that $\mathbb{Z}_6/H$ is isomorphic to $\mathbb{Z}_2$.

Recall Ex 1: $\mathbb{Z}_6 \cong \langle 2 \rangle \times \langle 3 \rangle$

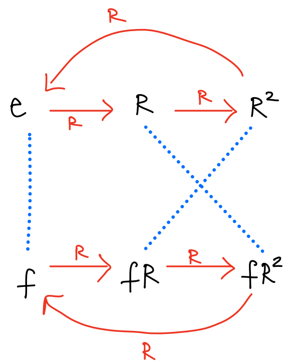
The quotient of S_3 by $\langle (123) \rangle$

Ex

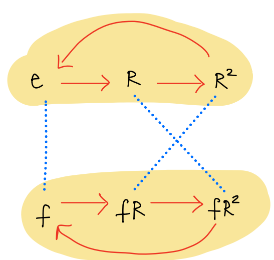
Cayley graph of $G = S_3$ with generators $R = (123)$, $f = (12)$



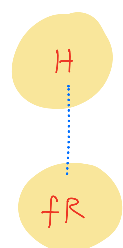
or



- Consider a normal subgroup $H = \langle (123) \rangle = \langle R \rangle$ which is isomorphic to $\mathbb{Z}_3$.
- There are two left cosets : $H = \{e, R, R^2\}$ and $fH = \{f, fR, fR^2\}$, so $G/H \cong \mathbb{Z}_2$.
- The following visualizes taking quotient of G by H:



Collapse each coset
into a single vertex



- Note S_3 is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_3$

Internal direct product

(Sec 9.2)

Back to Ex 1 and Ex 2

Def A group G is called the internal direct product of H and K, if:

1. H and K are normal subgroups of G
2. $G = HK$ (Recall: $HK \stackrel{\text{def}}{=} \{hk : h \in H, k \in K\}$)
3. $H \cap K = \{e\}$

Note If G, H, K satisfy all three conditions above, then by def G is the internal direct product of H and K, and G is naturally isomorphic (but not equal) to the external direct product of H and K.

(see Thm 9.27)

Ex 1 gives us the isomorphism $\mathbb{Z}_6 \cong \langle 3 \rangle \times \langle 2 \rangle$

Ex 2 —— " —— $D_6 \cong \langle f, R^2 \rangle \times \langle R^3 \rangle$

Example 9.24 The group $U(8)$ is the internal direct product of

$$H = \{1, 3\} \quad \text{and} \quad K = \{1, 5\}.$$

□

Example 9.25 The dihedral group D_6 is an internal direct product of its two subgroups

$$H = \{\text{id}, r^3\} \quad \text{and} \quad K = \{\text{id}, r^2, r^4, s, r^2 s, r^4 s\}.$$

(Ex 2)

It can easily be shown that $K \cong S_3$; consequently, $D_6 \cong \mathbb{Z}_2 \times S_3$. □

— end —