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Abstract Algebra Notes Week 5 Wed, Oct 2 2024

(Recall) Lemma (Book Lemma 6.3)

Let G be a group, H a subgroup, and $a, b \in G$.
Then the following conditions are equivalent.

① $aH = bH$ ② $Ha^{-1} = Hb^{-1}$ ③ $aH \subset bH$

④ $b \in aH$ ⑤ $a^{-1}b \in H$

(Recall) Thm (Lagrange) (Thm 6.10)

Let G be a finite group, H a subgroup.

Then $\frac{|G|}{|H|} = [G : H]$. In particular, $|H|$ divides $|G|$.

Cor 1 (Cor 6.11) Let G be a finite, $g \in G$. Then $|g|$ divides $|G|$.

Proof Exercise

Cor (Cor 6.12) If $|G| = p$ is prime, then

① G is cyclic

② any non-identity $g \in G$ is a generator.

(Every group of prime order is cyclic)

Proof

Since $p \geq 2$, there is some non-identity $g \in G$.

Then $|g|$ divides p by above Cor 1.

Since $g \neq e$, $|g| \neq 1$, so $|g| = p$.

Therefore $\langle g \rangle$ has order p, so $\langle g \rangle = G$. $\square$

Cor If $K \leq H \leq G$

(Cor 6:13) (K is a subgroup of H, and H is a subgroup of G),

then $[G:K] = [G:H][H:K]$

Proof $[G:K] = \dfrac{|G|}{|K|} = \dfrac{|G|}{|H|} \cdot \dfrac{|H|}{|K|} = [G:H][H:K]$ $\square$

$\boxed{\text{Dihedral groups (Sec 5.2)}}$ Let $n \geqslant 3$.

D_n = group of symmetries of a regular n-gon ($n \geqslant 3$)

Prop Let R denote the counterclockwise rotation by $\frac{2\pi}{n}$,

and f any reflection across a line of symmetry.

Then (1) $D_n = \{ Id, R, R^2, \ldots, R^{n-1},$ — rotations (including Id)

reflections/flips — $f, fR, fR^2, \ldots, fR^{n-1} \}$

where the items on the list are distinct.

The powers of R are rotations

fR^i are reflections.

(2) The order of each reflection is 2

The order of R is n, so $R^{-1} = R^{n-1}$

$R^{-i} = R^{n-i}$

Remark Part (1) of Prop above tells us that every element of D_n is a product of R and f.

We say D_n is generated by R and f.

Remark Since $|\text{reflection}| = 2$,

we have $Rf = (Rf)^{-1} = f^{-1}R^{-1} = fR^{-1}$

These are called "relations"

So $\boxed{Rf = fR^{-1}}$ (equivalently $fRf = R^{-1}$

Similarly, $\boxed{R^i f = fR^{-i}}$ (equivalently, $fR^i f = R^{-i}$)

Prop D_n is not abelian

Proof $f(fR) = (ff)R = R$ but $(fR)f = (R^{n-1}f)f = R^{n-1}$

Since $n \geqslant 3$, R and R^{n-1} are distinct.

Corollary D_n is not cyclic.

Generators and Cayley diagrams

Def Let G be a group, and let S be a subset of G.
We say that S is a <u>generating set of G</u> (or S is a <u>set of generators for G</u>) if every elt in G is a finite product of elts in S and their inverses. Notation: $G = \langle S \rangle$

Ex $D_n = \langle \text{Rot}(\frac{2\pi}{n}), f \rangle$ where f is any specific flip.

Ex $S_n = \langle \text{all transpositions} \rangle = \langle (12), (23), \ldots, (n-1, n) \rangle$

$S_4 = \langle (12), (23), (34) \rangle$, $S_3 = \langle (12), (23) \rangle$

Ex $\mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle = \langle 2, 3 \rangle = \langle 7, 12 \rangle = \langle 2, 3, 5 \rangle$

Def S is called <u>minimal</u> if no proper subset of S is a generating set of G.

different than <u>minimum</u>

Ex $\{2,3\}, \{2,3,5\}, \{1\}$ are all generating sets of $\mathbb{Z}$.
$\{2,3\}$ and $\{1\}$ are both minimal, and $\{2,3,5\}$ is <u>not</u> minimal.

Def Given a group G and a set of generators S, a <u>Cayley diagram</u> (or Cayley graph) consists of

(1) vertices : all elts of G

(2) colored (or labeled) arrows : all elts in generating set S

* Write $x \xrightarrow{h} y$ iff $xh = y$ for some $h \in S$

(applying arrow h means multiplying on the right)

Note Following an h-arrow backwards means multiplying on the right by h^{-1}:

$x \xrightarrow{h} y$ means $yh^{-1} = x$

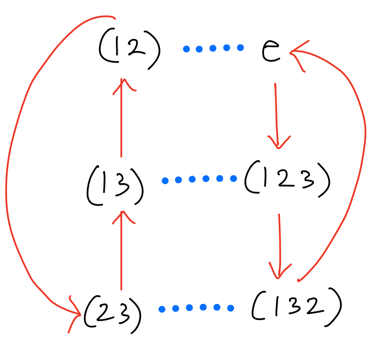
* If, in addition, h is its own inverse, then we have $xh = y$ iff $x = xhh = yh$

$$\begin{tikzcd} x \arrow[r, "h"] & y \arrow[l, bend left, "h"] \end{tikzcd} \quad \text{or} \quad x \overset{h}{\longleftrightarrow} y$$

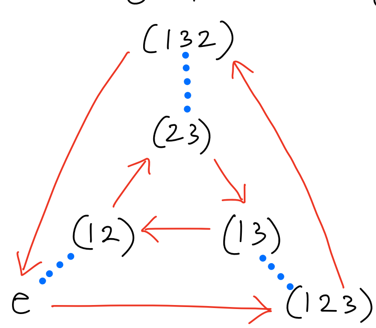
Our convention is to drop the tips on all these two-way arrows: $x \overset{h}{-\!\!-\!\!-\!\!-} y$

Ex Cayley diagram for generating set $\{(12), (123)\}$ of

$S_3 = \langle \overset{f}{(12)}, \overset{R}{(123)} \rangle$



The same graph rearranged:



Note A Cayley diagram can be used as a "group calculator". Start at e, then chase the sequence through the Cayley graph.

Ex What is $RRfRRRRf$ equal to? Ans: (123)
What is $RRfRRRRfR^{-1}$ equal to? Ans: e

Note We can use a Cayley diagram to "see" the cyclic subgroup $\langle x \rangle$ generated by an elt x.

Draw the path from e to x, then repeat the same path until we return to e.

Notation For this visual reason, we will refer to $\langle x \rangle$ as the <u>orbit</u> <u>of x</u>.

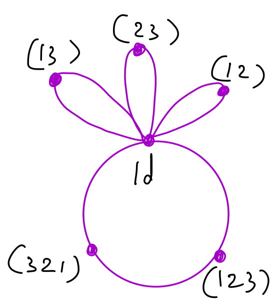
Ex The orbit of (132) is $\langle (132) \rangle = \langle R^2 \rangle = \{e, R^2, R\} = \{e, (132), (123)\}$

The orbit of (13) is $\langle (13) \rangle = \langle \underset{\underset{fR^2}{\parallel}}{Rf} \rangle = \{e, Rf\} = \{e, (13)\}$

We can visualize these orbits in an "orbit graph":

* Every elt will be part of at least one orbit
* Each cycle represents an orbit

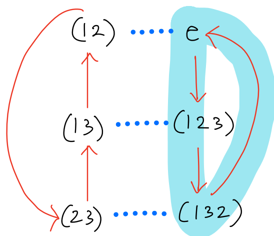
Ex The orbit graph of S_3:



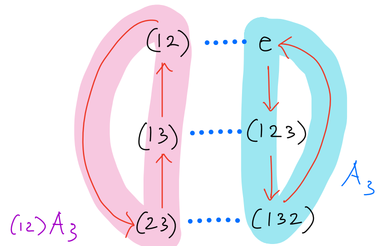
S_3 has five distinct orbits (including $\{Id\}$)

We can visualize subgroups and cosets

Ex the subgroup $A_3 = \langle (123) \rangle$ of S_3:



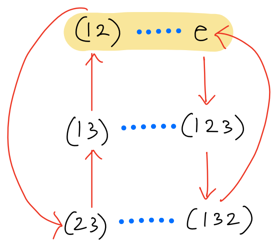
The two left cosets of A_3:



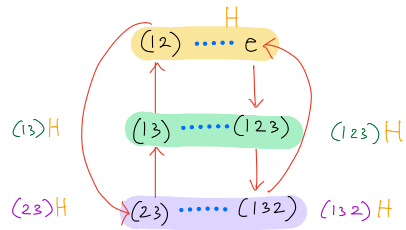
Prop If $[G:H] = 2$ (number of left cosets) then $gH = Hg$ for all $g \in G$.

Prof The left cosets $eH = H$ and gH partition G, & the right cosets $He = H$ and Hg partition G.
So $gH = Hg$.

the subgroup $\langle (12) \rangle$ of S_3:



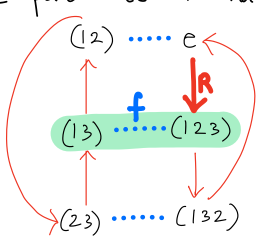
The three left cosets of H:



Note * To visualize the left coset gH in a Cayley graph, start from vertex g and follow all paths in H.

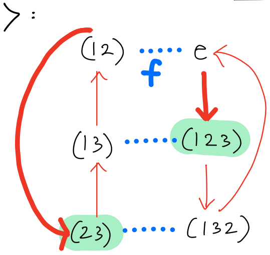
* Compare the left and right cosets $(123)H$ vs $H(123)$:

– Left coset $R\langle f \rangle$: the vertices the f arrows can reach after the path to R has been followed



– The left cosets of H look like copies of H

– Right coset $\langle f \rangle R$: the vertices the R arrows can reach from elts in $\langle f \rangle$:



– The right cosets are usually scattered

only because we adopted the convention that arrows mean right multiplication

Cayley diagrams of direct products

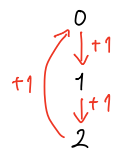
Let A, B be groups, and let e_A & e_B denote the identities of A & B (respectively).

Given a Cayley diagram of group A w/ generators $a_1, a_2, \dots, a_k$ and
a Cayley diagram of group B w/ generators $b_1, b_2, \dots, b_\ell$,
we can construct a Cayley diagram for direct product $A \times B$:

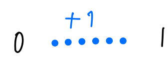
Vertices: (a,b) for each $a \in A$, $b \in B$
(often arranged in a rectangular grid)

Generators: $(a_1, e_B), \dots, (a_k, e_B)$ and $(e_A, b_1), \dots, (e_A, b_\ell)$.

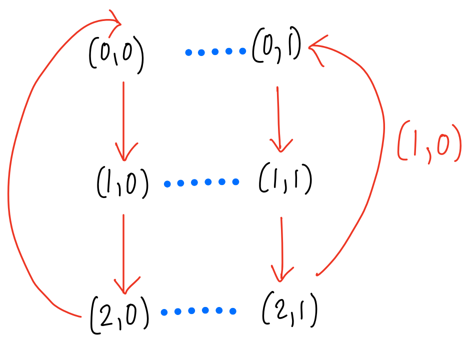
Ex: Cayley diagram for $\mathbb{Z}_3$ with generator 1:



Cayley diagram for $\mathbb{Z}_2$ with generator 1:



Cayley diagram for $\mathbb{Z}_3 \times \mathbb{Z}_2$ w/ generators $(0,1)$ and $(1,0)$:

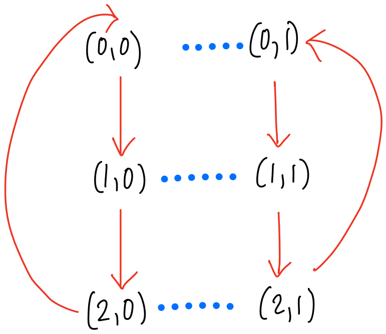


<u>Prop</u> If $H \le A$ and $K \le B$ then $H \times K$ is a subgroup of $A \times B$.

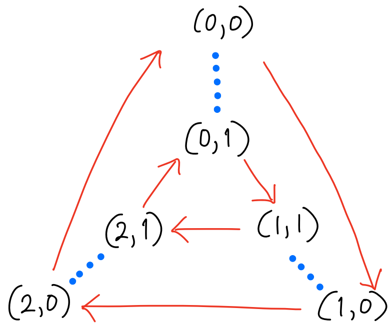
<u>Ex</u> $\mathbb{Z}_3 \times \{0\}$ is a subgroup of $\mathbb{Z}_3 \times \mathbb{Z}_2$

$\{0,2,4\} \times \{0,3\}$ is a subgroup of $\mathbb{Z}_6 \times \mathbb{Z}_6$

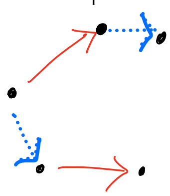
Ex Cayley diagram for generating set $\{(0,1), (1,0)\}$ of
$\mathbb{Z}_3 \times \mathbb{Z}_2 = \langle (1,0), (0,1) \rangle$

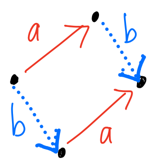


The same graph rearranged:

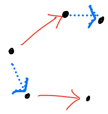


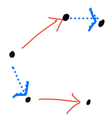
Note To check that a group G is abelian, it suffices to check that $ab=ba$ for all generators of G. (Why?)

① The pattern  never appears in the Cayley graph of an abelian group

② The pattern 

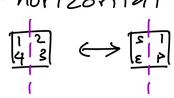
tells us $ab = ba$

Ex $\mathbb{Z}_3 \times \mathbb{Z}_2$ is abelian, and pattern  doesn't appear

but S_3 is not abelian, and pattern  does appear

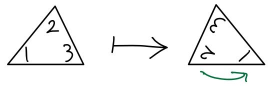
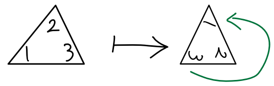
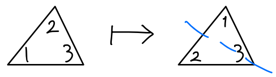
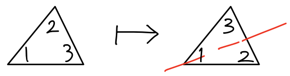
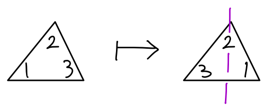
Group exercise: Draw the Cayley diagram for

* $\mathbb{Z}_4 \times \mathbb{Z}_2$ using $S = \{(1,0), (0,1)\}$
* D_4 using $S = \{R, f\}$ where $R = Rot(90°)$, f = horizontal flip



* $\mathbb{Z}_8$ using $S = \{3\}$

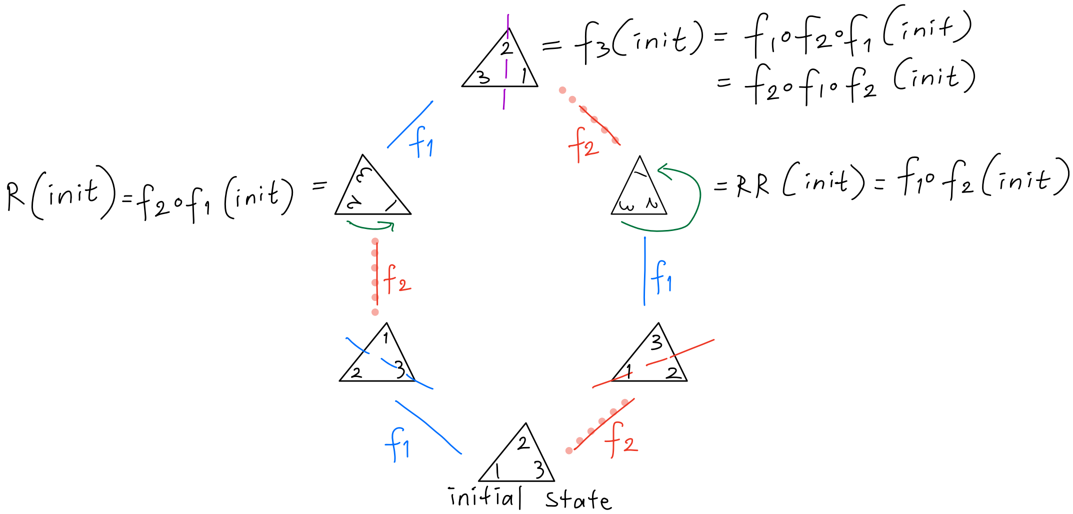
__Ex__ D_3 has 6 elts:

1) Identity
2) Counterclockwise rotation by $\frac{2\pi}{3}$ R: 
3) Counterclockwise rotation by $\frac{4\pi}{3}$ RR: 
4) Negative slope mirror flip f_1: 
5) Positive slope mirror flip f_2: 
6) Vertical mirror flip f_3: 

Compositions of f_1 and f_2 give us all other motions.

Another generating set for D_3 is $S = \{f_1, f_2\}$.

The Cayley graph for $\langle f_1, f_2 \rangle$ is below:



List of examples so far

	Group	order	Abelian?
always cyclic	$\mathbb{Z}_n$	n	yes
always cyclic	$\mathbb{Z}$	∞	yes
	$U(n)$	# of $k \in [n]$ relatively prime with n	yes
never cyclic	$\mathbb{R}$	∞	yes
not cyclic for $n \geqslant 3$	S_n	$n!$	No for $n \geqslant 3$
never cyclic	D_n $(n \geqslant 3)$	$2n$	No
not cyclic for $n \geqslant 4$	A_n	$\frac{n!}{2}$	No for $n \geqslant 4$
never cyclic	$GL_2(\mathbb{R})$	∞	No

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