Instruction: You can type this homework in the body of the email, or handwrite it, or use Overleaf. For those choosing to use Overleaf, the symbol for $\cap$ is \cap and the symbol of ϕ is \phi.

1 Second Isomorphism Theorem for abelian groups (+3 pts)

1.1 Statement

Copy the statement of Theorem 11.12 (Second Isomorphism Theorem) in Judson Section 11.2.

But assume G is abelian, so use + as the symbol for the binary operations. Instead of writing HN, you would write $H + N = \{h + n : h \in H, n \in N\}$.

1.2 Proof of Second Isomorphism Theorem for abelian groups

Write the proof of Theorem 11.12 (Second Isomorphism Theorem) in Judson Section 11.2. Follow the proof given in the book.

But assume G is abelian, so use + as the symbol for the binary operations. So you **will not include an explanation why** $H \cap N$ **is normal**, since every subgroup of an abelian group is normal. Also, the cosets in $H/(H \cap N)$ and in $(H + N)/N$ should be written with the + symbol.

2 Second Isomorphism Theorem for rings (+3 pts)

2.1 Statement

Copy the statement of Theorem 16.32 (Second Isomorphism Theorem for rings) in Section 16.3.

2.2 Proof of Second Isomorphism Theorem for rings

Write a proof of Theorem 16.32 (Second Isomorphism Theorem for rings).

For students enrolled in Math.4210, this last Question 2.2 will be graded by completion.
For students enrolled in Math.5210, this last question will be graded by correctness.

Hint: In Question 1.2, you defined a group homomorphism ϕ which you proved to be an surjective group homomorphism from I onto the quotient group $(I + J)/J$. To prove that ϕ is a surjective ring isomorphism, you only need to show that ϕ preserves the ring multiplication operation, that is, show that

$$\phi(rs) = \phi(r)\phi(s)$$

3 Example (+3 pts)

Consider the ring $R = \mathbb{Z}_{24}$ under the usual addition and multiplication mod 24.

1. List all elements in the principal ideal $I = \langle 4 \rangle = \{4r : r \in R\}$.

Solution: $I = \{0, 4, 8, 12, 16, 20\}$

List all elements in the principal ideal $J = \langle 6 \rangle$.

Solution: $J = \{0, 6, 12, 18\}$.

2. List all elements in $I + J$.

> **Solution:** $I + J = \{x + y : x \in I, y \in J\} = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22\}$.

3. List all elements in $I \cap J$.

> **Solution:** $I \cap J = \{0, 12\}$.

4. List all cosets in the quotient ring $(I + J)/J$. For each coset $x + J$, write all elements in the set $x + J$.

> **Solution:** Since there are 12 elements in $I + J$ and there are 4 elements in J, we know (from Lagrange's theorem) that there are 3 cosets.
>
> - $J = \{0, 6, 12, 18\}$
> - $2 + J = \{2, 2 + 6, 2 + 12, 2 + 18\} = \{2, 8, 14, 20\}$
> - $4 + J = \{4, 4 + 6, 4 + 12, 4 + 18\} = \{4, 10, 16, 22\}$

5. List all cosets in the quotient ring $I/(I \cap J)$. For each coset $x + (I \cap J)$, write all elements in the set $x + (I \cap J)$.

> **Solution:** Since there are 6 elements in I and there are 2 elements in $I \cap J$, we know (from Lagrange's theorem) that there are 3 cosets.
>
> An alternative way to realize that there are 3 cosets is by realizing that the number of elements in the quotient ring $I/(I \cap J)$ and in the quotient ring $(I + J)/J$ is the same, due to the Second Isomorphism Theorem for rings.
>
> - $I \cap J = \{0, 12\}$
> - $4 + (I \cap J) = \{4 + 0, 4 + 12\} = \{4, 16\}$
> - $8 + (I \cap J) = \{8 + 0, 8 + 12\} = \{8, 20\}$

6. Write down an explicit ring isomorphism (which is guaranteed to exist by the Second Isomorphism Theorem) from the quotient ring $(I + J)/J$ to the quotient ring $I/(I \cap J)$.

   Explicitly say which element in $(I + J)/J$ in sent to which element in $I/(I \cap J)$, for this specific example.

> **Solution:**
>
> $$J \mapsto (I \cap J)$$
> $$2 + J \mapsto 8 + (I \cap J)$$
> $$4 + J \mapsto 4 + (I \cap J)$$

4 Acknowledgements (+1 pt)

Write down everyone who contributed to your thought process. Write down Judson's textbook, class notes, and other written sources you used as well.