1 Ideal test (+1 pt)

Write down the two conditions in the "Ideal test" from Week 11 class notes. The first condition should be a sentence containing the word "additive" and "subgroup".

2 Principal ideal definition (+1 pt)

If R is a commutative wring with unity, write down what it means for a subset I of R to be a principal ideal of R.

(Note: if you can show that a subset I can be written this way, you do not need to check the two conditions in the "Ideal test" above.)

3 Isomorphism (+1 pt)

Consider the map $\varphi : \mathbb{C} \to \mathbb{C}$ defined by

$$\varphi(a + bi) = a - bi$$

(i) Prove that φ preserves multiplication.

(ii) Prove that φ is injective.

(See week 11 class notes)

4 Kernel and 1-to-1 homomorphism (+1 pt)

Let $\varphi : R \to S$ be a ring homomorphism. Prove the following: If $\ker \varphi = \{\mathbf{0}_R\}$, then φ is 1-to-1.

Hint: Write your own short proof, or see Week 9 class notes, Prop 5 (also Lemmas 3 and 4).

Solution: (Here I write a shorter self-contained proof)
Suppose $\ker \varphi = \{\mathbf{0}\}$. Let $\varphi(a) = \varphi(b)$. Then $\mathbf{0}_S = \varphi(b) - \varphi(a) = \varphi(b - a)$. So $b - a \in \ker \varphi = \{\mathbf{0}_R\}$, and so $b - a = \mathbf{0}_R$. Thus $b = a$.

5 The kernel of a ring homomorphism is an ideal (+1 pt)

Let $\varphi : R \to S$ be a ring homomorphism. Prove that the kernel of φ is an ideal of R.

(See week 11 class notes or Proposition 16.27 in Judson Section 16.3)

6 An ideal of the ring of integer polynomials? (+1 pt)

Let $\mathbb{Z}[x]$ denote the ring of all polynomials having integer coefficients. Consider the subset S of $\mathbb{Z}[x]$ of integer polynomials $f(x)$ such that $f(5) = 0$, that is, integer polynomials which have 5 as a root.

(a) What do the polynomials in S look like? Give some examples.

(b) Is S an ideal?

> **Solution:** Yes. See below.

(c) If S is a principal ideal, describe an element of S which generates S.

> **Solution:** $S = \langle x - 5 \rangle$

7 An ideal of the ring of real polynomials (+1 pt)

Consider the ring $\mathbb{R}[x]$ of polynomials with real coefficients, and let I denote the set of polynomials in $\mathbb{R}[x]$ with no constant term or term of degree 1. For example,

$$p(x) = \pi x^2 - ex^5 \in I,$$

but

$$q(x) = \pi - ex^5 \text{ and } r(x) = \pi x + x^2 \text{ are not in } I.$$

(a) Is I an ideal of $\mathbb{R}[x]$?

(b) Is I a principal ideal of $\mathbb{R}[x]$? If it is, describe an element of I which generates I.

Hint: See Example 17.19 in Judson Section 17.3

> **Solution:** $I = \langle x^2 \rangle$ is the principal ideal generated by x^2.

8 Ideals? (+2 pts)

Let $\mathbb{Z}[x]$ denote the ring of all polynomials having integer coefficients. Which of the following subsets of $\mathbb{Z}[x]$ are ideals? Answer **Yes** or **No**.

- If you answer No, provide a specific example of how the subset fails the absorbing property of an ideal or how the subset fails to be an additive subgroup of $\mathbb{Z}[x]$.

- If you answer Yes, explain why the absorbing property holds (you don't need to prove that the subset is an additive group).

(a) $S = \mathbb{Z}$, that is, all the constant polynomials in $\mathbb{Z}[x]$.

> **Solution:** No. The subset S is not an ideal of $\mathbb{Z}[x]$ because it fails the absorbing property. For example, $f(x) = 5$ is a polynomial in S and $g(x) = x^2$ is a polynomial in $\mathbb{Z}[x]$, but their product is $5x^2$ which is not in S.

(b) The set S of integer polynomials $f(x)$ such that $f(r) \geq 0$ for all real number r (when you graph the polynomial, the curve is always on or above the x-axis).

> **Solution:** No. The subset S is not an ideal because it fails the absorbing property. For example, $f(x) = x^2$ is a polynomial in S and $g(x) = -3$ is a polynomial in $\mathbb{Z}[x]$, but their product is $-3x^2$ which is not in S.

(c) The set S of integer polynomials $f(x)$ such that $f(1) \neq 0$, i.e. 1 is *not* a root of $f(x)$.

> **Solution:** No. The subset S is not an ideal because it fails the absorbing property. For example, $f(x) = x^2$ is a polynomial in S and $g(x) = (x - 1)$ is a polynomial in $\mathbb{Z}[x]$, but their product is $(x - 1)x^2$ which is not in S.

(d) The set S of integer polynomials $f(x)$ whose coefficients are all even integers.

> **Solution:** Yes, the absorbing property holds because any integer polynomial multiplied by a polynomial in S has even integer coefficients.

9 Acknowledgements (+1 pt)

Write down everyone who contributed to your thought process. Write down Judson's textbook, class notes, and other written sources you used as well.