1 Matrices with integer entries

Definition 1. Consider the set

$$\mathrm{Mat}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$$

of 2×2 matrices with integer entries. It forms a (non-commutative) ring with unity under the usual matrix addition and matrix multiplication. The unity of $\mathrm{Mat}_2(\mathbb{Z})$ is the identity matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and the zero element is the zero matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

For each of the following subsets S of $\mathrm{Mat}_2(\mathbb{Z})$, answer whether S is a subring of $\mathrm{Mat}_2(\mathbb{Z})$. (Answer **Yes/ No**)

If you claim S is not a subring, specify which subring conditions are not satisfied (S doesn't contain the zero element; S is not closed under ring addition; S is not closed under ring negation; S is not closed under ring multiplication)

(a) S is the subset of $\mathrm{Mat}_2(\mathbb{Z})$ consisting of matrices with determinant 1.

(b) S is the subset of $\mathrm{Mat}_2(\mathbb{Z})$ consisting of matrices with even entries.

(c) $S = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z} \right\}$ is the subset of upper-triangular matrices in $\mathrm{Mat}_2(\mathbb{Z})$.

(d) $S = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}$

2 Units

Definition 2. Let R be a ring which has unity denoted by the symbol $\mathbf{1}$. An element $u \in R$ is called a *unit* (also called an *invertible element*) if there exists $v \in R$ such that $uv = vu = \mathbf{1}$.

What are the units (if any) in the ring $\mathbb{Z}_{12}$?

(Hint: Example 3.11 in Section 3.2 Groups: Definitions and Examples computes the units for $\mathbb{Z}_8$.)

3 Fields

Use the definition and choose examples from class notes or Textbook's Section 16.2 or Section 16.1.

(a) Write the definition of a field.

(b) Give an example of an infinite field.

(c) Give an example of a field consisting of 29 elements.

4 Nilpotent elements

Definition 3. A ring element x is called *nilpotent* if $x^k = \mathbf{0}$ for some positive integer k.

What are the nilpotent elements (if any) in the ring $\mathbb{Z}_{12}$?

5 Question

(a) Let R be a ring with unity $\mathbf{1}$ and $x \in R$. Prove the following: if $x^3 = \mathbf{0}$ then $\mathbf{1} - x$ is a unit.

(Hint: Divide $1 - x^3$ by $1 - x$)

(b) Let R be a ring with unity $\mathbf{1}$ and let n be a positive integer. Is the following statement true or false? (Answer **True/ False**)

If $x \in R$ and $x^n = \mathbf{0}$, then $\mathbf{1} - x$ is a unit.

If you selected False, give a counterexample; otherwise, prove it in your own notebook.

6 Zero divisors

(a) Write down the definition of a *zero divisor.* (Use class notes or Textbook's Section 16.2 or Section 16.1)

(b) What are the zero divisors (if any) of the ring $\mathbb{Z}[i]$ of the Gaussian integers?

(See class notes/ Example 16.12 of Textbook's Sec 16.2)

(c) What are the zero divisors (if any) of the ring $\mathbb{Z}_{12}$?

7 Cancellation law in an integral domain

Suppose R is an integral domain and $x \in R$. If $x^2 = x$, what are the possible values of x?

(Hint: Review the "Cancellation law for integral domains' in Week 10 class notes or Proposition 16.15 in Textbook's Section 16.2)

8 Idempotents

Definition 4. Let R be a ring. An element x in R is called an *idempotent* if it satisfies $x^2 = x$.

What are the idempotents in $\mathbb{Z}_{12}$? (Hint: For each of the elements r in $\mathbb{Z}_{12}$, simply check whether $r^2 = r$.)

9 Characteristic of a ring

(a) Write down the definition of the *characteristic* of a ring. (Use class notes or Textbook's Section 16.2)

(b) Write down the statement and proof of Lemma 16.18 from Textbook's Section 16.2

(c) What is the characteristic of the ring $\mathbb{Z}[i]$ of the Gaussian integers?

(d) What is the characteristic of the ring $\mathbb{Z}_{12}$? What is the characteristic of the ring $\mathbb{Z}_{625}$?

10 Acknowledgements

Write down everyone who contributed to your thought process. Write down Judson's textbook, class notes, and other written sources you used as well.