MATH 4210/5210 ALGEBRA HOMEWORK 09

((SOLUTIONS))

Sample Homework Submission to help you with Question 1

((Delete or comment out this section before submitting the homework)) Given a group G, the *center of* G, denoted by $Z(G)$, is the subset of elements in G that commute with every element of G. In symbols,

$$Z(G) = \{a \in G : ax = xa \text{ for all } x \in G\}$$

Prove that $Z(G)$ is a subgroup of G.

Sample proof. To show that $Z(G)$ is a subgroup, we need to show the following: (1) The identity e of G is contained in $Z(G)$; (2) the subset $Z(G)$ is closed under the group operation of G; and (3) the subset $Z(G)$ is closed under taking inverses.

(1) The identity e is in $Z(G)$ because $ex = e = xe$ for all $x \in G$ by definition of the identity element.

(2) To show that the subset $Z(G)$ is closed under the group operation, we need to show that if $a, b \in Z(G)$ then $ab \in Z(G)$.

Suppose $a, b \in Z(G)$. (Our goal is to show that $ab \in Z(G)$, that is, $(ab)x = x(ab)$ for all $x \in G$.) Let $x \in G$. Then we have

$$\begin{aligned}(ab)x &= a(bx)\\ &= a(xb) \text{ since } b \text{ is in } Z(G)\\ &= (ax)b\\ &= (xa)b \text{ since } a \text{ is in } Z(G)\\ &= x(ab)\end{aligned}$$

This concludes the proof that $ab \in Z(G)$.

(3) To show that the subset $Z(G)$ is closed under taking inverses, we need to show that if $a \in Z(G)$ then its inverse a^{-1} is also in $Z(G)$.

Suppose $a \in Z(G)$. (Our goal is to show that $a^{-1} \in Z(G)$), that is, $a^{-1}x = xa^{-1}$ for all $x \in G$.) Let $x \in G$. Then we have

$$xa = ax,$$

since $a \in Z(G)$. Multiply on the left and on the right by a^{-1}:

$$\begin{aligned}a^{-1}(xa)a^{-1} &= a^{-1}(ax)a^{-1}\\ (a^{-1}x)(aa^{-1}) &= (a^{-1}a)(xa^{-1})\\ (a^{-1}x)e &= e(xa^{-1})\\ a^{-1}x &= xa^{-1},\end{aligned}$$

as needed.

□

Definitions and theorems relevant to this homework

Definition 1 (Product of subgroups)**.** Given two subgroups J and K of a group G, the set JK is defined to be

$$JK = \{jk : j \in H, k \in K\}$$

Remark 2 (Warning)**.** In general, JK is not a subgroup of G. For example, $J = \{Id, (12)\}$ and $K = \{Id, (23)\}$ are both subgroups of S_3, but their product is not a subgroup (for explanation why, see our Midterm Exam solutions, Problem 7, posted on Blackboard).

The following theorem is helpful for proving whether a subgroup is normal or not normal.

Theorem 3 (Theorem 10.3 of Judson)**.** Let H be a subgroup of G. Then the following are all equivalent.

(1) $gH = Hg$ for all $g \in G$ (that is, H is normal in G) ("left cosets are right cosets")
(2) $ghg^{-1} \in H$ for all $h \in H$ and all $g \in G$ ("closed under conjugation")
(3) $gHg^{-1} = H$ for all $g \in G$ ("only one conjugate subgroup")

1. Product of subgroups (+2 pts)

Suppose G is a group, and $H \leq G$ and $N \trianglelefteq G$. (Note the previous sentence says that H is any subgroup of a G, not necessarily normal in G, and N is a normal subgroup of G.) With these assumptions, prove that HN is a subgroup of G.

Proof. To show that HN is a subgroup, we need to show the following: (1) The identity e of G is contained in HN; (2) the subset HN is closed under the group operation of G; and (3) the subset HN is closed under taking inverses.

(1) $e = ee \in HN$ because $e \in H$ and $e \in N$

(2) Supose $a, b \in HN$. Then a and b can be written as $a = h_1 n_1$ and $b = h_2 n_2$ for some $h_1, h_2 \in H$ and $n_1, n_2 \in N$. Then

$$\begin{aligned} ab &= h_1 n_1 h_2 n_2 \\ &= h_1 h_2 n_3 n_2 \text{ for some } n_3 \in N, \text{ since } Nh_2 = h_2 N, \text{ due to the assumption that } N \text{ is normal} \end{aligned}$$

Therefore $ab \in HN$.

(3) Supose $a \in HN$. Then a and b can be written as $a = hn$ for some $h \in H$ and $n \in N$. Since N is normal and $h \in G$, we have $hN = Nh$. Thus there is $m \in N$ such that $a = hn = mh$. Then

$$a^{-1} = h^{-1} m^{-1}$$

Therefore $a^{-1} \in HN$. □

2. Intersection of subgroups (+2 pts)

Suppose again that G is a group, and $H \leq G$ and $N \trianglelefteq G$. Prove that $H \cap N$ is a *normal* subgroup of H.

Proof. ((Choose the most convenient of the three statements in Theorem 3 to prove that $H \cap N$ is a normal subgroup of H))

First, verify that $H \cap N$ is a subgroup of H.

To prove that $H \cap N$ is a normal subgroup, we will show that $hxh^{-1} \in H \cap N$ for all $h \in H$ and $x \in H \cap N$. Let $h \in H$ and $x \in H \cap N$. Then $hxh^{-1} \in H$ since h, x, h^{-1} are all in H. Also, $hxh^{-1} \in N$ since N is normal in G and $h \in H \subset G$. Combining the last two sentences, we have $hxh^{-1} \in H \cap N$. □

3. Computation, no work needed (+1 pt)

Let $G = S_4$, $H = \langle\ (1234)\ \rangle$, and $N = A_4$. On your own scratch paper, compute H, N, and $H \cap N$. Then use the fact that

$$H/(H \cap N) \cong (HN)/N \tag{1}$$

to quickly compute HN.

What is HN equal to?

$$H = \langle\ (1234)\ \rangle = \{(1234), (13)(24), (4321),\ Id\}$$

(Note that two of the permutations in H are odd, and two are even)

$$N = \{\text{all even permutations in } S_4\}$$

$$H \cap N = \{\text{the two even permutations in } H\} = \{(13)(24),\ Id\}$$

Since $|H| = 4$ and $|H \cap N| = 2$, the quotient group $H/(H \cap N)$ is of order 2. Using the fact $H/(H \cap N) \cong (HN)/N$ given to use above in (1), we have that the quotient group $(HN)/N$ is also of order 2. Since

$|N| = |A_4| = 12$, we know that $|HN|$ must be $2 \cdot 12 = 24$. Since HN is a subgroup of S_4, we conclude that $\boxed{HN \text{ is the entire group } S_4}$.

4. The First Isomorphism Theorem in words (+2 pts)

Explain the first isomorphism theorem in words to a classmate who has not seen it before. Do not use any math symbol. You might enjoy reading the blog post

The First Isomorphism Theorem, Intuitively by Tai-Danae Bradley (Math3ma).

((Write your explanation here))

5. True or false? (+1 pt)

(a) Given a homomorphism from G, we can construct from it a normal subgroup of G. True or false?
Given any homomorphism $f : G \to H$, we have proven that the kernel of f is a normal subgroup of G.

(b) Given a normal subgroup N of G, there is a homomorphism from G whose kernel is N. True or false?
The natural (or canonical) homomorphism $f : G \to G/N$ given by $x \mapsto xN$ has N as its kernel.

6. Counting abelian groups (+1 pt)

How many abelian groups of order $1176 = 2^3 \cdot 3 \cdot 7^2$ are there, up to isomorphism?
Hint: See Judson Example 13.5 for an example on how to use the Fundamental Theorem of Finite Abelian Groups to count isomorphism classes of abelian groups.

There are $\boxed{\text{six}}$ isomorphism classes of abelian groups of order 1176. Two of them are below:

$$\mathbb{Z}_8 \times \mathbb{Z}_3 \times Z_{49} \cong \mathbb{Z}_{24} \times Z_{49}$$

$$\mathbb{Z}_4 \times Z_2 \times \mathbb{Z}_3 \times Z_{49} \cong \mathbb{Z}_4 \times Z_6 \times Z_{49} \cong \mathbb{Z}_{12} \times Z_2 \times Z_{49}$$

There are four more classes not listed above.

7. Acknowledgements (+1 pt)

Write down everyone who helped you, including classmates who contributed to your thought process (either through sharing insights or through being a sounding board). Write down Judson's textbook, class notes, and other written sources you used as well.

((Fill in))