1 Normal subgroups of a direct product

Let $G = S_3 \times \mathbb{Z}_3$. Which of the following subgroups H of G are normal?

(a) $H = \{Id\} \times \{0\}$

(b) $H = \{Id\} \times \mathbb{Z}_3$

(c) $H = \{Id, (12)\} \times \{0\}$

(d) $H = S_3 \times \{0\}$

(e) $H = S_3 \times \mathbb{Z}_3$

Solution: All subgroups are normal except $H = \{Id, (12)\} \times \{0\} = \{\ (Id, 0),\ ((12), 0)\ \}$. The conjugate $((23), 0)((12), 0)((23), 0)^{-1} = ((23), 0)((12), 0)((23), 0)$ is equal to $((13), 0)$ which is not in H.

2 Simple subgroups

Definition 1. A group with no normal proper subgroups (except the trivial subgroup $\{e\}$) is called *simple*.

(a) What are the proper subgroups of $\mathbb{Z}_6$?

Solution:

- The trivial subgroup,
- $\langle 2 \rangle = \{2, 4, 0\}$, and
- $\langle 3 \rangle = \{3, 0\}$.

(b) Is $\mathbb{Z}_6$ a simple group? Answer **Yes/ No**

Solution: No, since it has no proper subgroup other than the trivial subgroup.

(c) If p is a prime number, what are the proper subgroups of $\mathbb{Z}_p$?

Solution: By Lagrange's theorem, since the only divisor of p is p and 1, the only proper subgroup is the trivial subgroup $\{0\}$.

(d) If p is a prime number, is $\mathbb{Z}_p$ a simple group? Answer **Yes/ No**

> **Solution:** Yes, since it has no proper subgroup other than the trivial subgroup.

(e) Is $S_3 \times \mathbb{Z}_3$ a simple group? Answer **Yes/ No**

> **Solution:** No, since we see from the previous question that it has normal subgroups.

Remark 2. Recall that the index $[S_n : A_n] = 2$, so A_n is a normal subgroup of S_n. This means that, for $n \geq 3$, the symmetric group S_n is *not* simple.

3 Alternating group A_4

Definition 3. Given a subgroup H of a group G and an element $x \in G$, the subset $xHx^{-1} = \{xhx : h \in H\}$ is called a *conjugate* of H.

Let $G = A_4 = \{$ the even permutations in S_4 $\}$.

(a) Let $H = \langle\ (123)\ \rangle = \{Id, (123), (321)\}$. Find all conjugates of the subgroup H in $G = A_4$. (Hint: H has four conjugates)

> **Solution:** The twelve permutations in A_4 are $\{e, (123), (321), (124), (421), (134), (431), (234), (432), (12)(34), (13)(24), (14)(23)\}$. To find all conjugates of H in A_4, compute xHx^{-1} for all $x \in A_4$.
>
> Note:
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> - It's possible that two different $x, y \in A_4$ give the same conjugate $xHx^{-1} = yHy^{-1}$.
> - If $x \in H$, then $xHx^{-1} = H$, so you don't need to do the conjugate computation for when $x \in H$.
> - All conjugates of a subgroup are also subgroups, so if xHx^{-1} is not a group, then you know you have made a computation mistake.
>
> The four conjugates of H are the following.
>
> - $eHe = \langle\ (123)\ \rangle = \boxed{\{e, (123), (321)\}}$
> - $(124)H(124)^{-1} = (124)H(421) = \{(124)e(421), (124)(123)(421), (124)(321)(421)\} = \boxed{\{e, (243), (342)\}}$
>   Also, $(134)H(134)^{-1} = (134)H(431) = \langle\ (234)\ \rangle$
> - $(234)H(234)^{-1} = (234)H(432) = \boxed{\{e, (134), (431)\} = \langle\ (134)\ \rangle}$
> - $(12)(34)H[(12)(34)]^{-1} = (12)(34)H(12)(34) = \{e, (12)(34)(123)(12)(34), (12)(34)(321)(12)(34)\} = \boxed{\{e, (421), (124)\}} = \langle(124)\rangle$

(b) Let $J = \langle\ (12)(34),\ (13)(24)\ \rangle = \{Id,\ (12)(34),\ (13)(24),\ (14)(23)\}$. Find all conjugates of the subgroup J in $G = A_4$.

Solution: For each $x \in A_4$, we have $xJx^{-1} = J$, so the only conjugate of J is itself.

(c) Is H a normal subgroup of A_4? Answer **Yes/ No**

Solution: No, since there are $x \in A_4$ such that $xHx^{-1} \neq H$.

(d) Is J a normal subgroup of A_4? Answer **Yes/ No**

Solution: Yes, since $xJx^{-1} = J$ for all $x \in A_4$.

(e) Is A_4 a simple group (according to Definition 1)?

Solution: No, since J is a nontrivial proper subgroup of A_4 which is normal.

Remark 4. A_5 is a simple group, and in general A_n is simple for $n \geq 5$. For more details, see Judson Section 10.2 The simplicity of the alternating group.