1 Normal subgroups of a direct product

Let $G = S_3 \times \mathbb{Z}_3$. Which of the following subgroups H of G are normal?

(a) $H = \{Id\} \times \{0\}$

(b) $H = \{Id\} \times \mathbb{Z}_3$

(c) $H = \{Id, (12)\} \times \{0\}$

(d) $H = S_3 \times \{0\}$

(e) $H = S_3 \times \mathbb{Z}_3$

2 Simple subgroups

Definition 1. A group with no normal proper subgroups (except the trivial subgroup $\{e\}$) is called *simple*.

(a) What are the proper subgroups of $\mathbb{Z}_6$?

(b) Is $\mathbb{Z}_6$ a simple group? Answer **Yes/ No**

(c) If p is a prime number, what are the proper subgroups of $\mathbb{Z}_p$?

(d) If p is a prime number, is $\mathbb{Z}_p$ a simple group? Answer **Yes/ No**

(e) Is $S_3 \times \mathbb{Z}_3$ a simple group? Answer **Yes/ No**

Remark 2. Recall that the index $[S_n : A_n] = 2$, so A_n is a normal subgroup of S_n. This means that, for $n \geq 3$, the symmetric group S_n is *not* simple.

3 Alternating group A_4

Definition 3. Given a subgroup H of a group G and an element $x \in G$, the subset $xHx^{-1} = \{xhx : h \in H\}$ is called a *conjugate* of H.

Let $G = A_4 = \{$ the even permutations in S_4 $\}$.

(a) Let $H = \langle\ (123)\ \rangle = \{Id, (123), (321)\}$. Find all conjugates of the subgroup H in $G = A_4$. (Hint: H has four conjugates)

(b) Let $J = \langle\ (12)(34),\ (13)(24)\ \rangle = \{Id,\ (12)(34),\ (13)(24),\ (14)(23)\}$. Find all conjugates of the subgroup J in $G = A_4$.

(c) Is H a normal subgroup of A_4? Answer **Yes/ No**

(d) Is J a normal subgroup of A_4? Answer **Yes/ No**

(e) Is A_4 a simple group (according to Definition 1)?

Remark 4. A_5 is a simple group, and in general A_n is simple for $n \geq 5$. For more details, see Judson Section 10.2 The simplicity of the alternating group.