MATH 4210/5210 ALGEBRA HOMEWORK 06

((SOLUTIONS))

INSTRUCTION: Complete Questions 1 through 4 below.

First Sample Answer

Consider the map $\psi : \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ defined by

$$\psi(m) = 3m$$

(a) Prove that ψ is a homomorphism.

Proof. Suppose $a, b \in \mathbb{Z}_{12}$. Then $\psi(a + b) = 3(a + b) = 3a + 3b = \psi(a) + \psi(b)$. □

(b) List the elements in the kernel of ψ.

Answer: The elements in $\ker\psi$ are the elements $a \in \mathbb{Z}_{12}$ such that $3a$ is congruent to 0 modulo 12, that is, $3a - 0$ is divisible by 12. So

$$\ker\psi = \{0, 4, 8\}$$

□

(c) Is ψ an isomorphism?

Answer: No, the function ψ is not an isomorphism. For example, we know ψ is not injective since $\psi(0) = 0 = \psi(4)$ although $0 \neq 4$ in $\mathbb{Z}_{12}$. □

Second Sample Answer

Let $H = \langle\ (12),\ (345)\ \rangle$ denote the subgroup of S_5 generated by (12) and (345).

Prove or disprove: There is an isomorphism from $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ to H.

Answer: True.

Proof: First, let's find all elements of $H = \langle (12), (345)\rangle$. By definition, H is the set of all products of $(12), (345)$, and their inverses. We find that $H = \{Id, (12)(345), (543), (12), (345), (12)(543)\}$, which is equal to

$$\{Id, c, c, c^2, c^3, c^4, c^5\} = \langle c\rangle, \text{ where } c = (12)(345).$$

So H is a cyclic group of order 6 with c as a generator. Every cyclic group of order 6 is isomorphic to $\mathbb{Z}_6$, so H is isomorphic to $\mathbb{Z}_6$ (meaning there exists an isomorphism between H and $\mathbb{Z}_6$).

We can now explicitly define an isomorphism from $\mathbb{Z}_6$ to H. Let $f : \mathbb{Z}_6 \to H$ be defined by

$$f(x) = c^x$$

□

Third Sample Answer

Let J denote the subgroup of S_5 generated by (13) and (24).

Prove or disprove: there is an isomorphism from $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ to $J = \langle\ (13),\ (24)\ \rangle$.

Answer: False.

Proof: First, let's find all elements of J. By definition, J is the set of all products of $(13), (24)$, and their inverses. We find that $J = \{Id, (13), (24), (13)(24)\}$. Observe that $x^2 = Id$ for all elements in J.

For the sake of contradiction, suppose there is an isomorphism $f : \mathbb{Z}_4 \to J$.

- Case $f(1) = Id$: Then $f(2) = f(1+1) = f(1)f(1) = Id \cdot Id = Id = f(1)$. Having $f(2) = f(1)$ means f is not injective.
- Case $f(1) \neq Id$: Then $f(1) = x$ where x is (13), (24), or (13)(24). Then

$$\begin{aligned} f(3) &= f(1+1+1) \\ &= f(1)f(1)f(1) \\ &= x^3 \\ &= x^2 x \\ &= x \text{ since we checked above that } x^2 = Id \\ &= f(1) \end{aligned}$$

  Having $f(3) = f(1)$ means f is not injective.

In both cases, f is not a bijection. So there is no isomorphism from $\mathbb{Z}_4$ to J. □

1. Question 1 (+3 pts)

Consider the map $\phi : \mathbb{C}^* \to \mathbb{C}^*$ defined by

$$\phi(z) = z^4$$

(a) Prove that ϕ is a homomorphism.

Proof. For all complex numbers $a, b \in \mathbb{C}^*$, we have

$$\begin{aligned} \phi(ab) &= (ab)^4 \\ &= a^4 b^4 \quad \text{since the multiplication of complex numbers is commutative} \\ &= \phi(a)\ \phi(b) \end{aligned}$$

□

(b) List the elements in the kernel of ϕ.

Note: Since 1 is the identity element in $\mathbb{C}^*$, the kernel of ϕ is $\ker \phi = \{z \in \mathbb{C}^* : \phi(z) = 1\}$.

Answer: The kernel of ϕ is the set of elements $z \in \mathbb{C}^*$ such that $\phi(z) = z^4 = 1$, so $\ker \phi$ is the set of the 4th roots of unity, $\{i, -1, -i, 1\}$. □

c.) Is ϕ an isomorphism? (If yes, prove that it is both surjective *and* injective; if no, prove that it's not injective *or* not surjective.)

Answer: No, it is not an isomorphism because ϕ is not injective. For example, $\phi(i) = 1 = \phi(1)$ but $i \neq 1$. □

2. Question 2 (+3 pts)

Prove or disprove: there is an isomorphism from $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ to $\mathbb{T}_4$, where

$\mathbb{T}_4$ is the set of all 4-th roots of unity $\{1, i, -1, -i\} = \{1, e^{i(\pi/2)}, e^{i(\pi/2)2}, e^{i(\pi/2)3}\}$.

Proof. Define a function $f : \mathbb{Z}_4 \to \mathbb{T}_4$ by

$$f(m) = i^m$$

Then $f(m+n) = i^{(m+n)} = i^m i^n = f(m)f(n)$ for all $m, n \in \mathbb{Z}_4$, so f is a homomorphism. It is a bijection since we have

$$f(0) = 1$$
$$f(1) = i$$
$$f(2) = -1$$
$$f(3) = -i$$

□

3. Question 3 (+3 pts)

Prove or disprove: there is an isomorphism from $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ to $U(8) = \{1, 3, 5, 7\}$.

Answer: False.

Proof: Suppose there is an isomorphism $f : \mathbb{Z}_4 \to U(8)$.

- Case $f(1) = 1$: Then $f(2) = f(1+1) = f(1)f(1) = 1 \cdot 1 = 1 = f(1)$. Having $f(2) = f(1)$ means f is not injective.
- Case $f(1) \neq 1$: Then $f(1) = x$ where $x = 3, 5$, or 7. Then

$$\begin{aligned} f(3) &= f(1+1+1) \\ &= f(1)f(1)f(1) \\ &= x^3 \\ &= x^2 x \\ &= x \text{ since } 3^2 = 1,\ 5^2 = 1, \text{ and } 7^2 = 1 \\ &= f(1) \end{aligned}$$

Having $f(3) = f(1)$ means f is not injective.

In both cases, f is not a bijection. So there is no isomorphism from $\mathbb{Z}_4$ to $U(8)$.

□

4. Acknowledgements (+1 pt)

Write down Judson's textbook, class notes, and other sources you used (if any). Also write down everyone who helped you, including classmates who contributed to your thought process (either through sharing insights or through being a sounding board), for example during the group quiz.

((FILL IN HERE))