1. (a) Consider the subgroup $H = \{(1), (1,2)\}$ of S_3. Are all left cosets of H the same as all right cosets of H?
(Possible answers: yes/no)

Solution: No. For example, the left coset $(123)H$ is not equal to the right coset $H(123)$. Another example that would work is $(13)H \neq H(13)$.

(b) Consider the subgroup $J = \{(1), (123), (132)\}$ of S_3. Are all left cosets of J the same as all right cosets of J?
(Possible answers: yes/no)

Solution: Yes. The left cosets are J itself and $\{(12), (13), (23)\}$. The right cosets are also J itself and $\{(12), (13), (23)\}$.
Note that this is a special case (where $n = 3$) of the more general situation which is asked about in the next question.

(c) Let $n \geq 3$. Consider the alternating subgroup A_n in the symmetric group S_n. Are all left cosets of A_n in S_n the same as the right cosets?
(Possible answers: yes/no)

Solution: Yes. Proof: There are exactly two left cosets of A_n in S_n. So the left coset xA_n which is not equal to A_n must equal the right coset which is not equal to A_n.

2. (a) What do the arrows in a Cayley diagram represent?

Solution: Each type of arrow (distinguished by color or label) represents a generator.

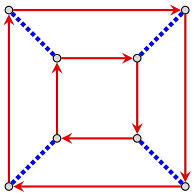
(b) What do the vertices in a Cayley diagram represent?

Solution: Each vertex represents an element in the group (so the number of vertices of the Cayley diagram is equal to the number of elements in the group).

3. Below are Cayley diagrams of eight different groups (none of them has the same group structure). For each Cayley diagram, determine whether the corresponding group is abelian or not abelian.
(Possible answers: abelian/not abelian)

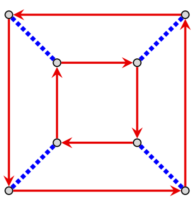
Solution:

The pattern on the left *never* appears in the Cayley graph for an abelian group, whereas the pattern on the right illustrates the relation $ab = ba$:

(A) 

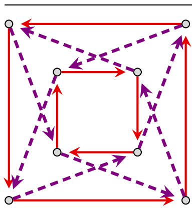
Solution: Abelian. This is the abstract Cayley graph for the direct product of $\mathbb{Z}_4 \times \mathbb{Z}_2$ with generators $(1, 0)$ (red solid arrow) and $(0, 1)$ (blue dotted arrow).

Note the solid arrow has order 4, and the dotted arrow has order 2.

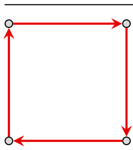
(B) 

Solution: Not abelian. This is the abstract Cayley graph for D_4 with generators rotation by $2\pi/4$ (red solid arrow) and a flip (blue dotted arrow).

Note the solid arrow has order 4, and the dotted arrow has order 2.

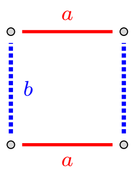
(C) 

Solution: Not abelian.

(D) 

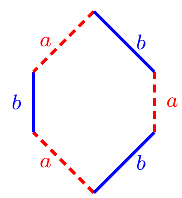
Solution: Abelian. Since there is only one generator, the forbidden pattern cannot appear.

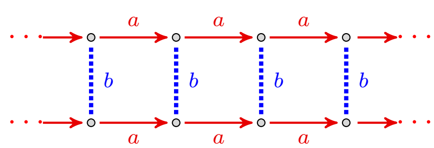
This is the abstract Cayley graph for $\mathbb{Z}_4$ with generator 1 or 3.

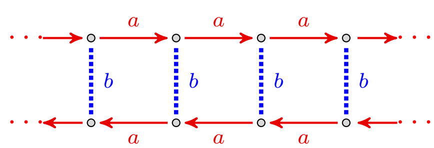
(E) 

Solution: Abelian. This is the abstract Cayley graph for the direct product of $\mathbb{Z}_2 \times \mathbb{Z}_2$ with generators $(1, 0)$ (red solid arrow) and $(0, 1)$ (blue dotted arrow).

Note the solid arrow has order 2, and the dotted arrow also has order 2.

(F) 

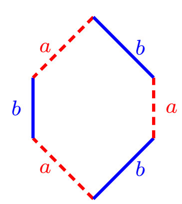
(G) 

(H) 

4. For each of the following group and its generating set, draw the Cayley graph on your own paper. Which Cayley graph from Question 3 is it?
(Possible answers: A, B, C, D, E, F, G, H)

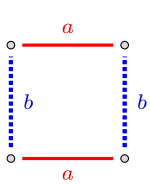
   (i) The group S_3 with minimal generating set the transpositions $(1\ 2)$ and $(2\ 3)$.

Solution: Let $a = (1\ 2)$ and $b = (2\ 3)$.



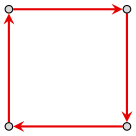
   (ii) The subgroup of S_4 generated by the transpositions $(1\ 2)$ and $(3\ 4)$.

Solution: Let $a = (1\ 2)$ and $b = (3\ 4)$.



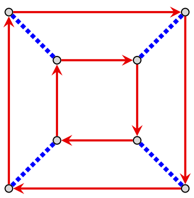
(iii) The subgroup of S_6 generated by the 4-cycle $(1\ 5\ 3\ 6)$.

Solution: Let the arrow be labeled by the generator $(1\ 5\ 3\ 6)$.



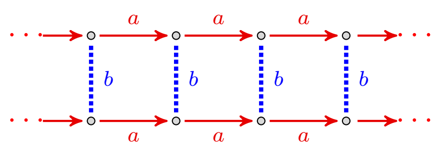
(iv) The direct product $\mathbb{Z}_4 \times \mathbb{Z}_2$ with two generators $(1, 0)$ and $(0, 1)$.

Solution: Let the solid (red) arrow denote the generator $(1, 0)$ which has order 4, and let the dotted (blue) arrow denote the generator $(0, 1)$ which has order 2.



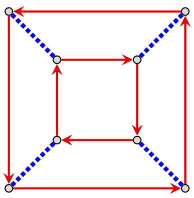
(v) The direct product $\mathbb{Z} \times \mathbb{Z}_2$ with generating set $\{(1, 0),\ (0, 1)\}$.

Solution: Let the solid (red) arrow denote the generator $a = (1, 0)$ which has infinite order, and let the dotted (blue) arrow denote the generator $b = (0, 1)$ which has order 2.



(vi) The dihedral group D_4 generated by R (the counterclockwise rotation by 90^o) and f (a flip along the vertical mirror).

Solution: Let the solid (red) arrow denote the generator R which has order 4, and let the dotted (blue) arrow denote the generator f which has order 2.



Solution:

1. 2 pts
2. 2 pts
3. 3 pts
4. 3 pts