MATH 4210/5210 ALGEBRA HOMEWORK 04

((SOLUTIONS))

Instruction: Complete the proofs and computation requested below in Questions 1 through 4.

Definition (from Judson Ch 6): Let H be a subgroup of a group G, and let g be an element of G. Then

$$gH = \{gh : h \in H\}$$

is the *left coset* of H containing g (or say "with representative g" instead of "containing g").

Similarly, $Hg = \{hg : h \in H\}$ is the *right coset* of H containing g.

1. Lemma 6.3 (3 pts)

The following is **Lemma 6.3 in Judson Chapter 6:**
Let H be a subgroup of a group G and suppose that $a, b \in G$. The following conditions are equivalent.

(1) $aH = bH$
(2) $Ha^{-1} = Hb^{-1}$
(3) $aH \subset bH$
(4) $b \in aH$
(5) $a^{-1}b \in H$

Prove that (4) implies (3). Hint: Partial proofs of this lemma are given in week 4 class notes and in solutions to week 5 practice.

Proof. We will prove that (4) implies (3). Suppose $b \in aH$. We will show that $aH \subset bH$.

[[Since $b \in aH$, we have that $b = ah$ for some $h \in H$. So $bh^{-1} = a$.

Let $x \in aH$. (We will show that $x \in bH$ also.) Then

$$\begin{aligned} x &= ak \text{ for some } k \in H \\ &= bh^{-1}k \text{ since } a = bh^{-1} \\ &\in bH \text{ since } h^{-1}k \in H \end{aligned}$$

So $aH \subset bH$.]] □

2. Conjugates and cosets (3 pts)

If $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$, prove that $gH = Hg$ for all $g \in G$ (that is, prove that the left cosets are identical to the right cosets).

Proof. Let g be an element in G.

First we show that $gH \subset Hg$. Let $x \in gH$. Then $x = gh_x$ for some $h_x \in H$, and so

$$\begin{aligned} xg^{-1} &= (gh_x)g^{-1} \\ &\in H \quad \text{by assumption} \end{aligned}$$

But this means

$$x = (xg^{-1})g \in Hg.$$

Therefore, $gH \subset Hg$.

Similarly, we can show that $Hg \subset gH$. [[Similarly, we can show that $Hg \subset gH$. Let $y \in Hg$. Then

$$y = h_y g$$

for some $h_y \in H$, and so

$$g^{-1}y = g^{-1}h_y g \in H$$

Then

$$y = g(g^{-1}y) \in gH$$

Therefore, $Hg \subset gH$.]] □

3. Computation (3 pts)

The converse of Question 2 is true, that is,

$$\text{if } gH = Hg, \text{ then } ghg^{-1} \in H \text{ for all } h \in H. \tag{3.1}$$

Let $G = S_9$. Consider the subgroup $H = \langle (2\,7) \rangle$, the cyclic group generated by the transposition $(2\,7)$.

3.1. Part 1.

Find a permutation $\sigma \in S_9$ such that $\sigma\ (2\,7)\ \sigma^{-1}$ is NOT equal to $(2\,7)$.

[[Many permutations will work. For example, any transposition which is not disjoint from $(2\,7)$ such as $\sigma = (2\,4)$ or $\sigma = (7\,8)$ will work]]

3.2. Part 2.

Are the left cosets of H and the right cosets of H in G all the same, or are some of them different? Why?

(Use (3.1) to answer this question. Don't attempt to list all cosets, since there are 181440 left cosets.)

[[No, for example $(2\,4)H = \{(2\,4),\ (2\,7\,4)\}$ but $H(2\,4) = \{(2\,4),\ (2\,4\,7)\}$.]]

4. Acknowledgements (1 pt)

Write down everyone who helped you, including classmates who contributed to your thought process (either through sharing insights or through being a sounding board). Write down Judson's textbook, class notes, and other sources you used as well.

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