MATH 4210/5210 ALGEBRA HOMEWORK 03

(SOLUTIONS)

1. Proof assignment (6 pts)

Let G be a group, and let $a, b \in G$. Prove that $|a| = |b^{-1}ab|$.

Proof. Proof (not assuming a has finite order): We first prove the statement for when a has a finite order.

Let n denote $|a|$. By definition (of the order of a group element), n is smallest positive integer such that $a^n = e$.

First, we will show that $|b^{-1}ab| \leq n$:

$$\begin{aligned}(b^{-1}ab)^n &= \underbrace{(b^{-1}ab)(b^{-1}ab)\dots(b^{-1}ab)}_{n \text{ copies}} \\ &= b^{-1}\underbrace{a\dots a}_{n \text{ copies}}b \\ &= b^{-1}a^n b \\ &= b^{-1}eb \text{ since } a^n = e \\ &= b^{-1}b \\ &= e\end{aligned}$$

We've shown that $(b^{-1}ab)^n = e$, so $|b^{-1}ab| \leq n$.

Next, we show $|b^{-1}ab| \geq n$:
Let k denote $|b^{-1}ab|$. Then

$$e = (b^{-1}ab)^k = b^{-1}a^k b.$$

Multiply on the left by b and on the right by b^{-1}, we have

$$\begin{aligned}beb^{-1} &= b(b^{-1}a^k b)b^{-1} \\ bb^{-1} &= (bb^{-1})(a^k)(bb^{-1}) \\ e &= a^k\end{aligned}$$

Since n is the smallest positive integer such that $e = a^n$, we can conclude that $k \geq n$. Therefore $|b^{-1}ab| \geq n$.

We will now prove the statement for when a has an infinite order. Suppose a has infinite order, that is, $a^n \neq e$ for all $n \in \mathbb{N}$. For the sake of contradiction, suppose $b^{-1}ab$ has a finite order k. Then $e = (b^{-1}ab)^k$. By the same argument as above, this implies $e = a^k$, which contradicts the fact that a has infinite order. □

2. Computation assignment (3 pts)

Consider the permutation $a = (153)(47)$ in S_{100}.

(a.) Compute the order of a.
Answer: $|a| = 6$ because $a \neq e$ and $a^2 = (135) \neq e$, $a^3 \neq e$, $a^4 \neq e$, $a^5 \neq e$, but $a^6 = e$.

(b.) Let $b = (19)(48)(57)(23)$. Compute the order of b.
Answer: $|b| = 2$ because $b \neq e$ but $b^2 = e$

(c.) *Without computing* bab, compute the order of bab.
Answer: Since $b^2 = e$, b is its own inverse, so $bab = b^{-1}ab$, and so $|bab| = |a| = 6$.

3. Acknowledgements (1 pt)

Write down everyone who helped you, including classmates who contributed to your thought process (either through sharing insights or through being a sounding board). Write down Judson's textbook, class notes, and other sources you used as well.