1. (a) Let $n > 1$. Let A_n and B_n denote the set of even permutations and the set of odd permutations, respectively. Define a map $f : A_n \to B_n$ by $f(\pi) = (1\,2)\pi$ for all $\pi \in A_n$.
Prove that this map is injective and surjective.

Solution: Use the same proof as the proof of Proposition 5.17 in Judson textbook here: http://abstract.ups.edu/aata/permute-section-permutation-definitions.html.

(b) Let H be a subgroup of a group G, and let $x \in G$. Define a bijective map f from H to xH.

Solution: Define
$$f\colon H \longrightarrow xH\,, \qquad \text{by } f(h) = xh$$
for all $h \in H$.

(c) Show that this map is surjective.

Solution: Suppose $b \in xH$. Then by definition of left coset, $b = xh$ for some $h \in H$. Let $a := h$. Then $f(a) = xa = xh = b$, as needed.

(d) Suppose G is a non-abelian group of order 1000 and H is a subgroup of order 20. Let x be an element of G which is not in H.
(i) How many elements are in the left coset xH?
(ii) How many elements are in the right coset Hx?

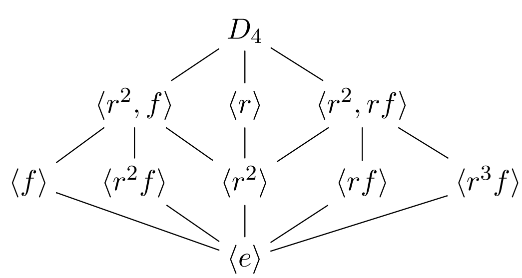
Solution: (i-ii)The size of every left coset (and also right coset) is the same as the size of H, so the answer is 20 for both questions.

(iii) How many left cosets of H are there?

Solution: (iii) By Lagrange's Theorem, there are $1000/20 = 50$ left cosets of H.

2. (a) Find all subgroups of D_4, and arrange them in a subgroup lattice. Moreover, label each edge between $K \leq H$ with the index, $[H : K]$.

Solution: The subgroup lattice of D_4 is shown below. The label on each edge is 2.



(b) Is $f\langle r\rangle = \langle r\rangle f$? What about other left and right cosets of $\langle r\rangle$? Prove your answer.

Solution: Yes, $x\langle r\rangle = \langle r\rangle x$ for all $x \in D_4$. First, we see that the group $\langle r\rangle$ has order 4. We know that the group D_4 has order 8. By Lagrange's theorem, we get that $[D_4 : \langle r\rangle] = 8/4 = 2$. We've seen in class that this implies that the left cosets of $\langle r\rangle$ and the right cosets of $\langle r\rangle$ coincide.

(c) Is the left coset $r^3 f\langle r^2, f\rangle$ equal to the right coset $\langle r^2, f\rangle r^3 f$?

Solution: Yes. Similar explanation as the previous part.

3. For each statement below, determine if it is true or false. Prove your answer.

(a) If the order of a group G is infinite (that is, if there are infinitely many elements in G), then the order of every non-identity $x \in G$ is also infinite.

Solution: False. Consider the infinite direct product $\mathbb{Z} \times D_3$. There are infinitely many elements in this group because $\mathbb{Z}$ is infinite, but the order of the element $(0, f)$, where f is one of the flips, is 2.

(b) Every cyclic group is abelian.

Solution: True.
Possible explanation 1: Every cyclic group is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_n$.
Possible explanation 2: A cyclic group G is a group which can be generated by only one element, so $G = \langle r\rangle$ for some $r \in G$. If $x, y \in G$, then $x = r^k$ and $y = r^\ell$ for some $k, \ell \in \mathbb{Z}$. So $xy = r^k r^\ell = r^{k+\ell} = r^\ell r^k = yx$.

(c) Every abelian group is cyclic.

Solution: False. Proof: A possible counterexample is V_4 or $\mathbb{Z}_2 \times \mathbb{Z}_2$ which is not cyclic (since it's a group of order 4 which is not isomorphic to $\mathbb{Z}_4$). It requires at least two generators.

(d) Every dihedral group is abelian.

Solution: False. Proof: The dihedral group D_3 of order 6 is not abelian, for example, rotation by 120^o followed by a flip is not the same as the same flip followed by a rotation by 120^o.

(e) Every symmetric group is not abelian.

Solution: False. The symmetric group $S_2 = \{Id,\ (12)\}$ on two objects is cyclic and therefore abelian.

(f) There is a cyclic group of order 100.

Solution: True. Proof: Take the additive group $\mathbb{Z}_{100}$, or the multiplicative subgroup of the 100-th roots of unity in $\mathbb{C}^*$.

(g) There is a symmetric group of order 100

Solution: False. The number 100 is not equal to any factorial. Check that $4! = 24 < 100 < 5! = 120$.

(h) If some pair of distinct, non-identity elements in a group commute, then the group is abelian.

Solution: False. In D_3, the elements R and R^2 commute, but D_3 is not abelian.

(i) If every pair of elements in a group commute, the group is cyclic.

Solution: False. The group V_4 is not cyclic, but every pair of elements commutes.

(j) If every pair of elements in a group commute, the group is abelian.

Solution: True, by definition.

4. (a) Is there a dihedral group of order 27?

Solution: No. A dihedral group has n reflections and n rotations (for some positive integer n), so the order of a dihedral group is even.

(b) If an alternating group A_n has order M, what order does the symmetric group S_n have?

Solution: The order of S_n is $2M$, since we've seen that there is a bijection between the set of even permutations and the set of odd permutations and even permutations of S_n.

5. For each part below, compute the orbit of the element in the group. Your answer should be a list of elements from the group that ends with the identity.

(a) The element R^2 in the group D_{10}

Solution: $\{R^2, R^4, R^6, R^8, e\}$

(b) The element 10 in $\mathbb{Z}_{16}$

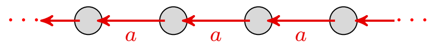
Solution: $\{10, 4, 14, 8, 2, 12, 6, 0\}$

(c) The element 25 in the group $\mathbb{Z}_{30}$

Solution: $\{25, 20, 15, 10, 5, 0\}$

6. Recall that $\mathbb{Z}$ is a group under the operation of ordinary addition.

(a) Create a Cayley diagram for it.

> **Solution:** If we choose a minimal generating set $\{1\}$, we have the following (where $a = +1$):
>
> 
>

(b) Is it abelian?

> **Solution:** Yes, it is a cyclic group, since it can be generated by the element 1 or -1.

(c) Give a minimal generating set consisting of more than one element.

> **Solution:** For example, $\{2, 3\}$ or $\{7, 12\}$ would work.

7. (a) Is there a group (of order larger than 1) in which no element (other than the identity) is its own inverse?

> **Solution:** Yes. For example, the cyclic group of order 3. You can observe this from the multiplication table.

(b) Is there a group (of order larger than 3) in which no element (other than the identity) is its own inverse?

> **Solution:** Yes. For example, the cyclic group of order 5. Every non-identity element has order 5, by Lagrange's Theorem.

(c) Find a group (of order larger than 1) such that there is only one solution to the equation $x^2 = e$, that is, the solution $x = e$, or explain why no such group exists.

> **Solution:** The groups in the solutions to parts (a), (b) would work.

(d) Find a group that has exactly two solutions to the equation $x^2 = e$, or explain why no such group exists.

> **Solution:** The cyclic group of order 4 generated by r. The two solutions are $x = e$ and $x = r^2$.

(e) Find a group with more than 2 solutions to the equation $x^2 = e$, or explain why no such group exists.

> **Solution:** The Klein-4 group $\langle a, b \rangle$ with minimal generating set $\{a, b\}$. There are four solutions, $x = e, x = a, x = b$, and $x = ab$. You can observe this from the multiplication table, or consider the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and check that all four elements satisfy the equation $x^2 = e$.

(f) Find a group with at least two elements in it, and only one solution to the equation $x^3 = e$ (that is, the solution $x = e$) or explain why no such group exists.

> **Solution:** The groups $\mathbb{Z}_2, \mathbb{Z}_4$, and V_4 would work.

(g) Find a group that has more than two solutions to the equation $x^3 = e$, or explain why no such group exists.

> **Solution:** In the cyclic group $\mathbb{Z}_3$, every element satisfies the equation $x^3 = e$.

(h) There are 2 non-isomorphic groups of order 6. What are their names? Specify which, if any, are abelian.

> **Solution:** One is non-abelian, the Dihedral group D_3 which is isomorphic to the symmetric group S_3. The other is the cyclic group $\mathbb{Z}_6$, which is abelian.

(i) Suppose m is a positive integer. If there exists only one group of order m, to what family must this group belong? Why?

> **Solution:** For each positive integer k, we have the cyclic group $\mathbb{Z}_k$ is a group. Since there exists only one group of order m, this group must belong to the family of cyclic groups.

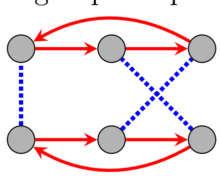
8. (a) If H is a subgroup of G and $a \in G$, then a left coset aH is ... [give the definition]

> **Solution:** the set $\{ah : h \in H\}$

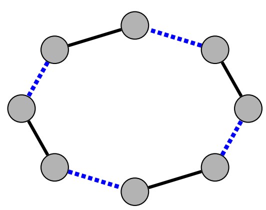
(b) The *index* $[G : H]$ of a subgroup $H \leq G$ is [give a definition, not a theorem!] ...

> **Solution:** ... the number of left cosets of H.

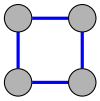
9. Determine whether each of the following diagrams are Cayley diagrams. If the answer is "yes," say what familiar group it represents, including the generating set. If the answer is "no," explain why.

(a) 

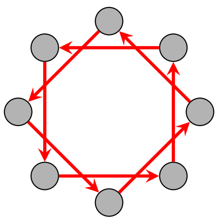
> **Solution:** Yes.
> This is the Cayley diagram of D_3 with generating set $\{R, f\}$, where R is a rotation by $2\pi/3$ and f is any flip.
> It could also be the Cayley diagram of S_3 with generating set $\{(123), (12)\}$.

(b) 

Solution: Yes. This is the Cayley diagram of the Dihedral group D_4 with minimal generating set f, g, where f and g are reflections with respect to mirrors 45^o apart.

(c) 

Solution: No. There is only one type of arrow, which means that there is only one generator. This arrow is double-sided, which means that this generator is of order 2. If this is the Cayley diagram of a group, the group should have order 2, not 4.

(d) 

Solution: No. There is only one type of arrow, which means that there is only one generator. This arrow has order 4 because we see that four arrows form a 4-cycle. If this is the Cayley diagram of a group, the group should have order 4, not 8.

10. Answer the following questions about permutations and the symmetric group.

(a) Write as a product of disjoint cycles (read from right to left as usual):
$(1\ 5\ 2)\,(1\ 2\ 3\ 4)\,(1\ 3\ 5) =$
$(1\ 3\ 5)\,(1\ 2\ 3\ 4)\,(1\ 5\ 2) =$

Solution: $(1\ 5\ 2)\,(1\ 2\ 3\ 4)\,(1\ 3\ 5) = (1\ 4\ 5)(2\ 3)$
$(1\ 3\ 5)\,(1\ 2\ 3\ 4)\,(1\ 5\ 2) = (1)(2)(3\ 4)(5) = (3\ 4)$

(b) Write $(1\ 2\ 3\ 4)$ as a product of *transpositions* (i.e., 2-cycles). Read from right to left as usual.

Solution: (14)(13)(12) or (12)(24)(23) or (23)(31)(34) or (34)(24)(14) (or other options)

(c) What is the *inverse* of the element $(1\ 3\ 2\ 6)\,(4\ 5)$ in S_6?

Solution: (45)(1623)

(d) The *order* of an element $g \in G$ is equal to the order (number of elements) of $\langle g \rangle$, the group generated by g. When the order is finite, it is also the minimum positive integer k such that $g^k = e$. What is the order of the element $(1\ 2\ 3\ 6)\,(4\ 5\ 7)$ in S_7?

Solution: The order is 12 since $[(1\ 2\ 3\ 6)\,(4\ 5\ 7)]^i \neq id$ for $i = 1, 2, ..., 11$ and $[(1\ 2\ 3\ 6)\,(4\ 5\ 7)]^{12} = id$

(e) Find an element of order 20 in S_9.

Solution: (1 2 3 4 5) (6 7 8 9)

Theorem 1. Let H be a subgroup of G. Then the following are all equivalent.

(i) The subgroup H is called *normal* in G, that is, $gH = Hg$ for all $g \in G$; ("left cosets are right cosets");

(ii) $ghg^{-1} \in H$ for all $h \in H, g \in G$; ("closed under conjugation").

(iii) $gHg^{-1} = H$ for all $g \in G$; ("only one conjugate subgroup")

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11. (a) Consider the subgroup $H = \{(1), (1, 2)\}$ of S_3. Is H normal?

Solution: No, you can check that $(123)H$ is not equal to $H(123)$.

Another example that would work is $(13)H \neq H(13)$.
A possibly faster way to determine this is to see that $(13) = (23)(12)(23)^{-1}$ and $(23) = (13)(12)(13)^{-1}$ are conjugate to (12), but they are not in H, hence failing part (ii) of the above theorem for being normal.

(b) Consider the subgroup $J = \{(1), (123), (132)\}$ of S_3. Is J normal?

Solution: Yes, there is only other left coset of J (other than J itself), and there is only other other right coset of J (other than J), so they must be the same.
This satisfies part (i) of the above theorem, Theorem 1, for being normal.

(c) Consider the subgroup $H = \langle (1234) \rangle$ of S_4. Is H normal?

Solution: No. For example, the 4-cycle (1324) is a conjugate of (1234) but it is not in H.

(d) Let $n > 2$. Is A_n a normal subgroup of S_n?

Solution: Yes. Proof: There are exactly two left cosets of A_n in S_n. So the left coset xA_n which is not equal to A_n must equal the right coset which is not equal to A_n.

(e) Consider a mystery subgroup K of $\mathbb{Z}_5 \times \mathbb{Z}_8$. Is K normal?

Solution: Every subgroup of an abelian group is normal, so K is normal.

12. Let H be a subgroup of G. Given two fixed elements $a, b \in G$, define the sets

$$aHbH := \{ah_1bh_2 : h_1, h_2 \in H\} \qquad \text{and} \qquad abH := \{abh : h \in H\}.$$

(a) Prove that *if* H is normal then $aHbH \subset abH$.

Solution: To show $aHbH \subset abH$, let $h_1, h_2 \in H$. We need to show that ah_1bh_2 can be written as abh for some $h \in H$. Since H is normal in G, the left coset bH is equal to the right coset Hb. Hence we can write h_1b as bh_3 for some $h_3 \in H$, so $ah_1bh_2 = abh_3h_2$, which is in abH since $h_3h_2 \in H$.

(b) Prove that the statement is false if we remove the "normal" assumption. That is, give a specific G and H and $a, b \in G$ such that $aHbH$ is not a subset of abH.

Solution: Possible proof: Let $G = D_3$, let $H = \langle f \rangle$. But $rfre = rfr = f$, which is in $rHrH$ but not in $r^2H = \{r^2, r^2f\}$, so $rHrH \neq r^2H$.

Try to come up with a similar proof but using S_3.

Possible scratch work (thought process):

Let $G = D_3$ (because every group with order 5 or lower is abelian). To come up with a counterexample, I have to make sure to pick a non-normal subgroup H (since the statement is true if H is a normal subgroup), so I can pick one of the subgroups which is generated by exactly one reflection, $\langle f \rangle$ or $\langle rf \rangle$ or $\langle r^2f \rangle$.

I pick $H := \{e, f\}$. To come up with a counterexample, I have to make sure to pick $a, b \notin H$ (otherwise the statement would be true).

First, I try $a = r$ and $b = r$, and I check whether $aHbH = abH$.

I first compute abH (because I see abH has a simpler definition that the other set).

Computing abH, I get $abH = r^2H = \{r^2, r^2f\}$.

Now, I try to find an element in $aHbH = rHrH$ which is not in r^2H. Since H has only two elements, to compute all elements of $aHbH$ I just need to compute $aebe$, $aebf$, $afbe$, and $afbf$. But I see that the first two are in abH by Definition of abH, so I will only check the last two elements.

I try $afbe = rfr = f$, which is not in abH. This example would be enough to show that $rHrH \neq rrH$.

(You can also try $a = b = rf$, or $a = r$ and $b = rf$, and see what happens.)

(c) In class, we proved that multiplication of cosets of N is well-defined if N is a normal subgroup.
Give an example where "multiplication" of cosets is not well-defined. That is, give a group G and a subgroup H where $a_1H = a_2H$ and $b_1H = b_2H$ but $a_1b_1H \neq a_2b_2H$.

Solution: You can use the same G and H as in the previous question. Just make sure your a_1, a_2, b_1, b_2 are not in H.

Another possible example is the following:

Consider the symmetric group S_3 and let $J := \langle (1\ 2) \rangle$.
Then the three left cosets of J are:

(a) $J = \{e, (1\ 2)\}$,

(b) $(132)J = (13)J = (1\ 3), (1\ 3\ 2)\}$, and

(c) $(1\ 2\ 3)J = (2\ 3)J = \{(2\ 3), (1\ 2\ 3)\}$.

Take $a_1 := (132)$, $a_2 := (13)$,
$b_1 := (123)$, and $b_2 := (23)$.

Then $a_1 b_1 J = (132)(123)J = eJ = J$, but $a_2 b_2 J = (13)(23)J = (123)J \neq J$.

13. (a) Given two groups A and B, what is the definition of the set $A \times B$?
  (b) Review the binary operation on $A \times B$
  (c) What is the identity element of $A \times B$?

Solution: $(1_A, 1_B)$, where 1_A is the identity element of A, and 1_B is the identity element of B.

  (d) If $(a, b) \in A \times B$, what is the inverse $(a, b)^{-1}$ equal to?

Solution: (a^{-1}, b^{-1})

  (e) Assume that neither of A and B is the trivial group. Prove that these four subgroups are normal in $A \times B$:

$$\{e_A\} \times \{e_B\}, \qquad A \times \{e_B\}, \qquad \{e_A\} \times B, \qquad A \times B$$

14. (a) True or false? The order of the group D_n is the same as the order of the group $\mathbb{Z}_2 \times \mathbb{Z}_n$.

Solution: True, the order is $2n$ for both.

  (b) True or false? The group D_n is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_n$.

Solution: False. If $n \geq 3$, the Dihedral group D_n is non-abelian, but $\mathbb{Z}_2 \times \mathbb{Z}_n$ is.

  (c) True or false? The group $\mathbb{Z}_{14}$ is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_7$.

Solution: True.
A possible proof: Note that $\mathbb{Z}_2 \times Z_7$ can be generated by the single element $(1, 1) \in \mathbb{Z}_2 \times \mathbb{Z}_7$ which has order 14, the least common multiple of 2 and 7. So $\mathbb{Z}_2 \times Z_7$ is a cyclic group of order 14.

  (d) True or false? The group $\mathbb{Z}_{16}$ is isomorphic to the group $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Solution: False. The group $\mathbb{Z}_{16}$ contains an element of order 16, that is, the number 1. Every element in the group $\mathbb{Z}_4 \times \mathbb{Z}_4$ has order 1, 2, or 4, so it cannot be generated by just one element; thus $\mathbb{Z}_4 \times \mathbb{Z}_4$ is not a cyclic group.

(e) Is $\mathbb{Z}_{12}$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_6$?

Solution: No. The group $\mathbb{Z}_2 \times \mathbb{Z}_6$ has no element of order 12.

(f) Which direct product is isomorphic to $\mathbb{Z}_{12}$?

Solution: The direct product $\mathbb{Z}_4 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_{12}$, since it can be generated by the element $(1, 1)$ which has order 12.

15. Let H be a subgroup of G.

(a) What does the notation G/H mean?

Solution: The set of all left cosets of H in G, that is, $\{xH \mid x \in G\}$.

(b) When is G/H a group?

Solution: When H is a normal subgroup of G.

(c) If G/N is a quotient group, what is the binary operation of the quotient group G/N?

Solution: $(aN)(bN) := abN$.

(d) Consider the symmetric group S_3 and a subgroup $H := \langle(1\ 2)\rangle$. Is the set $S_3/\langle(1\ 2)\rangle$ a quotient group? Prove your answer. If it is a quotient group, what is it isomorphic to?

Solution: No, $S_3/\langle(1\ 2)\rangle$ is not a quotient group because H is not normal in S_3.
A possible proof: The left coset $(123)\langle(1\ 2)\rangle = \{(23), (123)\}$ and the right coset $\langle(1\ 2)\rangle(123) = \{(13), (123)\}$ are not equal.
Another way to see that H is not normal is to recall that there are conjugates of (12) which are not in H, namely, (13) and (23).

(e) Consider the symmetric group S_3 and a subgroup $J := \langle(1\ 2\ 3)\rangle$. Is S_3/J a quotient group? Prove your answer. If it is a quotient group, what is it isomorphic to?

Solution: Yes, S_3/J is a quotient group because J is normal in S_3.

A possible proof: Since the order of S_3 is 6 and the order of J is 3, there are two left cosets of J. Hence the left coset of J (which is not J itself) must be equal to the right coset of J (which is not equal to J itself).

The quotient froup S_3/J is isomorphic to $\mathbb{Z}_2$ since there are two left cosets of J in S_3.

16. The following are all normal subgroups of D_4:

    (a) The trivial subgroup $\{e\}$,
    (b) the only normal subgroup of order 2, $\langle r^2 \rangle$,
    (c) all the subgroups of order 4: $\langle r \rangle$, $\langle r^2, f \rangle$, $\langle r^2, rf \rangle$, and
    (d) D_4 itself.

    For each N above, what familiar group is D_4/N isomorphic to?

    Solution: The only one that we have to compute carefully is $D_4/\langle r^2 \rangle$. We know that the number of cosets in $D_4/\langle r^2 \rangle$ is 4, but there are two groups of order 4 (up to isomorphism), so let's list the cosets in $D_4/\langle r^2 \rangle$: $\langle r^2 \rangle$, $r\langle r^2 \rangle$, $f\langle r^2 \rangle$, and $rf\langle r^2 \rangle$.

    By inspection, we see that each element (each coset) in $D_4/\langle r^2 \rangle$ has order 2, so this quotient group must be isomorphic to V_4, and not to $\mathbb{Z}_4$.

    Final answer:
    $D_4/\{e\} \cong D_4$
    $D_4/\langle r^2 \rangle \cong V_4$,
    For each subgroup H of order 4, we have $D_4/H \cong \mathbb{Z}_2$, and
    $D_4/D_4 \cong \{e\}$.

17. Let H be a subgroup of G, and consider the subset of G denoted by

$$\mathrm{Nor}_G(H) = \{g \in G : gH = Hg\} = \{g \in G : gHg^{-1} = H\}.$$

    Note: this set $\mathrm{Nor}_G(H)$ is often called the *normalizer of H in G*; it is the set of elements in G that "vote" in favor of H's normality.

    (a) Prove that $\mathrm{Nor}_G(H)$ is a subgroup.
    (b) What is the smallest that $\mathrm{Nor}_G(H)$ can be?

    Solution: H

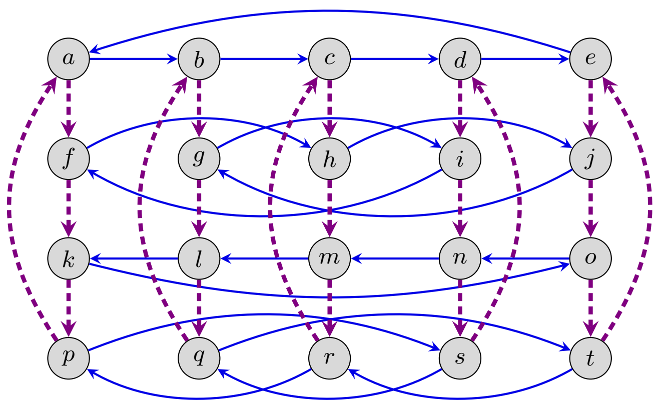
    (c) What is the largest $\mathrm{Nor}_G(H)$ can be?

    Solution: G

    (d) When does the latter happens?

> **Solution:** $\text{Nor}_G(H) = G$ if and only if H is normal.

18. Let G be the group whose Cayley diagram is shown below, and suppose e is the identity element. Consider the subgroups $A = \langle a \rangle = \{a, b, c, d, e\}$ and $J = \langle j \rangle = \{e, j, o, t\}$.



Carry out the following steps for both of the subgroups A and J. List the cosets element-wise.

(a) Write G as a disjoint union of the left cosets of A. Write G as a disjoint union of the left cosets of J.

(b) Write G as a disjoint union of the right cosets of A. Write G as a disjoint union of the right cosets of J.

(c) Use your coset computation to immediately compute the normalizer of the subgroup. Based on the computation for the normalizer, what you can say about this subgroup?

> **Solution:**
> $\text{Nor}_G(A) = G$, which means $A \trianglelefteq G$.
> $\text{Nor}_G(J) = J$, which means that J is as "unnormal" as possible.

(d) If G/A is a group, perform the quotient process and draw the resulting Cayley diagram for G/A.

> **Solution:** The quotient group G/A is isomorphic to $\mathbb{Z}_4$.

If G/J is a group, perform the quotient process and draw the resulting Cayley diagram for G/J.

> **Solution:** Since J is not normal, the set A/J is not a group.

19. The *center* of a group G is the set

$$Z(G) = \{z \in G \mid gz = zg, \text{ for all } g \in G\} = \{z \in G \mid gzg^{-1} = z, \text{ for all } g \in G\}.$$

It is a subgroup of G.

a. Prove that $Z(G)$ is normal in G by showing $ghg^{-1} \in H$ for all $h \in H, g \in G$ ("closed under conjugation").

Solution: Suppose $g \in G$. By Theorem 1, it is sufficient to show that $gzg^{-1} \in Z(G)$ for all $z \in Z(G)$. But, if $z \in Z(G)$, then $gzg^{-1} = z \in Z(G)$ for all $g \in G$.

b. Compute the center of $\mathbb{Z}_6$.

Solution: $\mathbb{Z}_6$ is abelian, so the entire group is the center.

c. Compute the center of D_4.

Solution: The center of D_4 is $\langle R^2 \rangle$. Reason: the half circle rotation commutes with every reflection (and every rotation). A different rotation does not commute with a reflection (for example, f). None of the reflections commutes with R.

d. Compute the center of D_5.

Solution: The center of D_5 is the trivial group. Reason: None of the rotations commutes with f. None of the reflections commutes with R.

e. Consider the group A_3 of even permutations. Compute the center of A_3.

Solution: A_3 is abelian, and therefore the center of A_3 is the entire group A_3.

To see why A_3 is abelian, notice that A_3 is a cyclic group of order 3, since it can be generated by the 3-cycle (123).

Another way to see that A_3 is abelian, is to compute its order which is $3!/2 = 3$. We've seen that every group of order 3 (or any prime number) is cyclic.

f. Consider the group A_n of even permutations, where $n \geq 4$. Prove that (1 2 3) is not in the center of A_n by producing another even permutation which does not commute with (1 2 3).

Solution: The element (2 3 4) works. (2 3 4)(1 2 3) = (12)(34)
(1 2 3)(2 3 4) = (13)(24)

g. Let $n \geq 4$. Prove that (1 2)(3 4) is not in the center of A_n.

Solution: For example, you can show that the element (1 2 3) does not commute with (1 2)(3 4).

h. Compute the center of A_4

Hint: A non-identity permutation in S_4 is an even permutation if and only of its cycle notation is of the form $(ab)(cd)$ or (abc). (Make sure you can prove this!)

Do $(ab)(cd)$ and (abc) commute?

Solution: Answer: The answer is the trivial group.
Reason: The permutations (abc) and $(ab)(cd)$ do not commute.
$(abc)(ab)(cd) = (a)(bdc)$ and $(ab)(cd)(abc) = (acd)(b)$.

i. Compute the center of S_4.
Hint: Every non-identity permutation in S_4 can be written in the form (ab), (abc), $(abcd)$, and $(ab)(cd)$. Can you find a permutation that does not commute with (ab)? With $(abcd)$?

j. Compute the center of S_2.

Solution: This group is abelian, so the center is the entire group.

k. Prove that "the center of a direct product is the direct product of the centers", that is, $Z(A \times B) = Z(A) \times Z(B)$.

Solution: First, it is clear that $Z(A \times B) \supset Z(A) \times Z(B)$.
To show that $Z(A \times B) \subset Z(A) \times Z(B)$, let $(z_1, z_2) \in Z(A \times B)$. Then, by definition, $(z_1, z_2)(g_1, g_2) = (g_1, g_2)(z_1, z_2)$ for all $g_1 \in A$, $g_2 \in B$. This means that $(z_1 g_1, z_2 g_2) = (g_1 z_1, g_2 z_2)$ for all $g_1 \in A$, $g_2 \in B$. In other words, $z_1 g_1 = g_1 z_1$ and $z_2 g_2 = g_2 z_2$ for all $g_1 \in A$, $g_2 \in B$, so $z_1 \in Z(A)$ and $z_2 \in Z(B)$.

20. Notation/Definition: Let G be a group and $x \in G$.

- The *conjugacy* class of x is the set $\mathrm{cl}_G(x) := \{gxg^{-1} \mid g \in G\}$.
- Let $Z(G)$ be the set $\{z \in G \mid gz = zg \text{ for all } g \in G\}$.

(a) Prove that $\mathrm{cl}_G(x) = \{x\}$ if and only if $x \in Z(G)$.

(b) Suppose N is a normal subgroup of G. Prove that if $x \in N$, then $\mathrm{cl}_G(x) \subset N$.

Solution: Let $x \in N$. Since N is normal in G, we have $gxg^{-1} \in N$ for all $g \in G$. Thus, $\mathrm{cl}_G(x) := \{gxg^{-1} : g \in G\} \subset N$.

21. You can use the following fact.

Proposition 1. For any $\sigma \in S_n$, we have $\sigma\ (a_1\ a_2\ \ldots\ a_k)\ \sigma^{-1} = (\,\sigma(a_1)\ \sigma(a_2)\ \ldots\ \sigma(a_k)\,)$.

(a) Let x be a k-cycle. Prove that $y \in S_n$ is conjugate to x iff y is a k-cycle.

Solution: By Proposition 1, every pair of k-cycles are conjugate.

(b) Prove that (12) and (14) in S_6 are conjugate by finding a permutation $p \in S_6$ such that $p^{-1}(12)p = (14)$.

(c) List all permutations in S_4 which are conjugate to (1234). Use the fact from part (a).

Solution: The answer is (1234), (1432), (1243), (1342), (1324), (1423). Explanation: The permutations which are conjugate to (1234) in S_4 are all the 4-cycles.

Proposition 2. Let $f : G_1 \to G_2$ be a homomorphism of groups. Then

(a) If e_1 is the identity of G_1, then $f(e_1)$ is the identity of G_2.
(b) For any element $g \in G_1$, $f(g^{-1}) = [f(g)]^{-1}$.
(c) If H_1 is a subgroup of G_1, then $f(H_1)$ is a subgroup of G_2.
(d) (i) If H_2 is a subgroup of G_2, then $f^{-1}(H_2) = \{g \in G_1 : f(g) \in H_2\}$ is a subgroup of G_1.
(ii) Furthermore, if H_2 is normal in G_2, then $f^{-1}(H_2)$ is normal in G_1.

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22. Prove all parts of Proposition 2.

> **Solution:** Proofs given under the Proposition 11.4 of Judson: http://abstract.ups.edu/aata/homomorph-section-group-homomorphisms.html

23. (a) Let $f : G_1 \to G_2$ be a homomorphism of groups. Prove that the kernel of f is a normal subgroup of G_1.

> **Solution:** Note that $\{e_2\}$ is a normal subgroup of the codomain G_2. By part (d)(ii) of above, $f^{-1}(\{e_2\})$ is normal.
>
> See also proof of Theorem 11.5 of Judson: http://abstract.ups.edu/aata/homomorph-section-group-homomorphisms.html which is given in the paragraph between Proposition 11.4 and Theorem 11.5

(b) Let $f : G \to H$ be a group homomorphism. Show that if $\ker(f)$ is the trivial group $\{1_G\}$ then f is injective.

24. (a) Let $f : G_1 \to G_2$ be a *surjective* homomorphism. Prove that, if $N \lhd G_1$, then $f(N)$ is normal in G_2.

> **Solution:** We need to show that $x_2\, f(N)\, x_2^{-1} \subset f(N)$ for all $x_2 \in G_2$.
> Suppose $x_2 \in G_2$. Since f is surjective, there is $x_1 \in G_1$ such that $f(x_1) = x_2$. Note that every element in $f(N)$ can be written as $f(n)$ for some $n \in N$. Then
>
> $$\begin{aligned} x_2\, f(n)\, x_2^{-1} &= f(x_1)\, f(n)\, f(x_1)^{-1} \\ &= f(x_1)\, f(n)\, f(x_1^{-1}) \\ &= f(x_1\, n\, x_1^{-1}) \in f(N) \end{aligned}$$
>
> since $x_1\, n\, x_1^{-1} \in N$ (because N is normal in G_1).

(b) If $f : G_1 \to G_2$ is a homomorphism and N is a normal subgroup of G_1, is it possible that $f(N)$ is not normal in G_2? If so, give an example.

> **Solution:** It is possible. Note that your example would require a non-surjective homormophism.
> For example, consider $f : \mathbb{Z}_2 \to S_3$ defined by $f(1) = (1\ 2)$ and let $N = \mathbb{Z}_2$. Then $f(N) = \langle(1\ 2)\rangle$, which is not normal in S_3.
> To see that $\langle(1\ 2)\rangle$ is not normal in S_3, check that the left coset and the right coset with coset representative $(1\ 2\ 3)$ are not equal.

25. Let $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ be the map given by $\phi(n) = 7n$ for $n \in \mathbb{Z}$. Find the kernel and the image of ϕ.

> **Solution:** The kernel of ϕ is the trivial subgroup $\{0\}$. The image of ϕ is $7\mathbb{Z}$, the subgroup of all integer multiples of 7.

26. Consider the group homomorphism $f : (\mathbb{R}, +) \to (\mathbb{C}^*, \times)$ defined by

$$f(\theta) = \cos\theta + i \sin\theta.$$

(a) Find the kernel of f and the image of f.

> **Solution:** The kernel is the subgroup $\{2\pi k : k \in \mathbb{Z}\}$ of $(\mathbb{R}, +)$. The image is the circle subgroup $\{x \in \mathbb{C}^* : |x| = 1\} = \{a + ib \in \mathbb{C}^* : \sqrt{a^2 + b^2} = 1\}$ of $\mathbb{C}^*$.

(b) Give an isomorphism (bijective group homomorphism) from the kernel of f to $(\mathbb{Z}, +)$.

> **Solution:** Let f send each $2\pi k \in \ker f$ to $k \in \mathbb{Z}$.

27. Let G be a group and let g be some element in G. Consider the group homomorphism $f : \mathbb{Z} \to G$ given by

$$f(n) = g^n.$$

(a) If the order of g is infinite, what is the kernel of f? Justify.

> **Solution:** The kernel is the trivial subgroup $\{0\}$ of $\mathbb{Z}$.

(b) If the order of g is finite, say m, what is the kernel of f? Justify.

> **Solution:** The kernel is the subgroup $m\mathbb{Z} = \{mk : k \in \mathbb{Z}\}$ of $\mathbb{Z}$.

28. True or false? Given two groups A and B, there exists a homomorphism from A to B. Prove your answer.

> **Solution:** True, the map $f : A \to B$ where $f(x) = e_B$ for all $x \in A$ (where e_B is the identity element in B) is a homomorphism. The kernel of this f is A, and the image of f is the trivial subgroup $\{e_B\}$ of B.

29. Given a homomorphism $f : G \to H$ define a relation $\sim$ on G by $a \sim b$ if $f(a) = f(b)$ for $a, b \in G$.

(a) Show that this relation is an equivalence relation.

(b) Describe the equivalence classes. How many classes are there?

> **Solution:** Check the three properties of being an equivalence relation.
>
> Description of the equivalence classes: Each element $h \in f(G)$ determines an equivalence class of the form $\{g \in G \mid f(g) = h\}$. The equivalence classes are in bijection to the elements of $f(G)$. There are as many equivalence classes as the number of elements in $f(G)$.
>
> Extra information: If f is a surjection, then there is a bijection between the equivalence classes and H.