1. (a) Let $n > 1$. Let A_n and B_n denote the set of even permutations and the set of odd permutations, respectively. Define a map $f : A_n \to B_n$ by $f(\pi) = (1\,2)\pi$ for all $\pi \in A_n$.
   Prove that this map is injective and surjective.
   (b) Let H be a subgroup of a group G, and let $x \in G$. Define a bijective map f from H to xH.
   (c) Show that this map is surjective.
   (d) Suppose G is a non-abelian group of order 1000 and H is a subgroup of order 20. Let x be an element of G which is not in H.
   (i) How many elements are in the left coset xH?
   (ii) How many elements are in the right coset Hx?
   (iii) How many left cosets of H are there?

2. (a) Find all subgroups of D_4, and arrange them in a subgroup lattice. Moreover, label each edge between $K \leq H$ with the index, $[H : K]$.
   (b) Is $f\langle r\rangle = \langle r\rangle f$? What about other left and right cosets of $\langle r\rangle$? Prove your answer.
   (c) Is the left coset $r^3 f\langle r^2, f\rangle$ equal to the right coset $\langle r^2, f\rangle r^3 f$?

3. For each statement below, determine if it is true or false. Prove your answer.
   (a) If the order of a group G is infinite (that is, if there are infinitely many elements in G), then the order of every non-identity $x \in G$ is also infinite.
   (b) Every cyclic group is abelian.
   (c) Every abelian group is cyclic.
   (d) Every dihedral group is abelian.
   (e) Every symmetric group is not abelian.
   (f) There is a cyclic group of order 100.
   (g) There is a symmetric group of order 100
   (h) If some pair of distinct, non-identity elements in a group commute, then the group is abelian.
   (i) If every pair of elements in a group commute, the group is cyclic.
   (j) If every pair of elements in a group commute, the group is abelian.

4. (a) Is there a dihedral group of order 27?
   (b) If an alternating group A_n has order M, what order does the symmetric group S_n have?

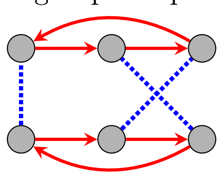
5. For each part below, compute the orbit of the element in the group. Your answer should be a list of elements from the group that ends with the identity.
   (a) The element R^2 in the group D_{10}
   (b) The element 10 in $\mathbb{Z}_{16}$
   (c) The element 25 in the group $\mathbb{Z}_{30}$

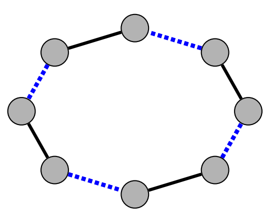
6. Recall that $\mathbb{Z}$ is a group under the operation of ordinary addition.
   (a) Create a Cayley diagram for it.
   (b) Is it abelian?
   (c) Give a minimal generating set consisting of more than one element.

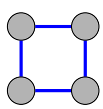
7. (a) Is there a group (of order larger than 1) in which no element (other than the identity) is its own inverse?
   (b) Is there a group (of order larger than 3) in which no element (other than the identity) is its own inverse?
   (c) Find a group (of order larger than 1) such that there is only one solution to the equation $x^2 = e$, that is, the solution $x = e$, or explain why no such group exists.
   (d) Find a group that has exactly two solutions to the equation $x^2 = e$, or explain why no such group exists.
   (e) Find a group with more than 2 solutions to the equation $x^2 = e$, or explain why no such group exists.
   (f) Find a group with at least two elements in it, and only one solution to the equation $x^3 = e$ (that is, the solution $x = e$) or explain why no such group exists.
   (g) Find a group that has more than two solutions to the equation $x^3 = e$, or explain why no such group exists.
   (h) There are 2 non-isomorphic groups of order 6. What are their names? Specify which, if any, are abelian.
   (i) Suppose m is a positive integer. If there exists only one group of order m, to what family must this group belong? Why?

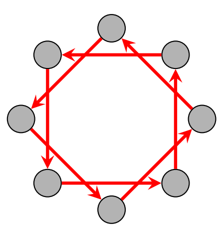
8. (a) If H is a subgroup of G and $a \in G$, then a left coset aH is ... [give the definition]
   (b) The *index* $[G : H]$ of a subgroup $H \leq G$ is [give a definition, not a theorem!] ...

9. Determine whether each of the following diagrams are Cayley diagrams. If the answer is "yes," say what familiar group it represents, including the generating set. If the answer is "no," explain why.

(a) 

(b) 

(c) 

(d) 

10. Answer the following questions about permutations and the symmetric group.

   (a) Write as a product of disjoint cycles: $(1\ 5\ 2)\,(1\ 2\ 3\ 4)\,(1\ 3\ 5) =$
   (b) Write $(1\ 2\ 3\ 4)$ as a product of *transpositions* (i.e., 2-cycles).
   (c) What is the *inverse* of the element $(1\ 3\ 2\ 6)\,(4\ 5)$ in S_6?

(d) The *order* of an element $g \in G$ is equal to the order (number of elements) of $\langle g \rangle$, the group generated by g. When the order is finite, it is also the minimum positive integer k such that $g^k = e$. What is the order of the element $(1\ 2\ 3\ 6)\,(4\ 5\ 7)$ in S_7?

(e) Find an element of order 20 in S_9.

Theorem 1. Let H be a subgroup of G. Then the following are all equivalent.

(i) The subgroup H is called *normal* in G, that is, $gH = Hg$ for all $g \in G$; ("left cosets are right cosets");

(ii) $ghg^{-1} \in H$ for all $h \in H, g \in G$; ("closed under conjugation").

(iii) $gHg^{-1} = H$ for all $g \in G$; ("only one conjugate subgroup")

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11. (a) Consider the subgroup $H = \{(1), (1,2)\}$ of S_3. Is H normal?
(b) Consider the subgroup $J = \{(1), (123), (132)\}$ of S_3. Is J normal?
(c) Consider the subgroup $H = \langle (1234) \rangle$ of S_4. Is H normal?
(d) Let $n > 2$. Is A_n a normal subgroup of S_n?
(e) Consider a mystery subgroup K of $\mathbb{Z}_5 \times \mathbb{Z}_8$. Is K normal?

12. Let H be a subgroup of G. Given two fixed elements $a, b \in G$, define the sets

$$aHbH := \{ah_1bh_2 : h_1, h_2 \in H\} \qquad \text{and} \qquad abH := \{abh : h \in H\}.$$

(a) Prove that *if* H is normal then $aHbH \subset abH$.
(b) Prove that the statement is false if we remove the "normal" assumption. That is, give a specific G and H and $a, b \in G$ such that $aHbH$ is not a subset of abH.
(c) In class, we proved that multiplication of cosets of N is well-defined if N is a normal subgroup.
Give an example where "multiplication" of cosets is not well-defined. That is, give a group G and a subgroup H where $a_1H = a_2H$ and $b_1H = b_2H$ but $a_1b_1H \neq a_2b_2H$.

13. (a) Given two groups A and B, what is the definition of the set $A \times B$?
(b) Review the binary operation on $A \times B$
(c) What is the identity element of $A \times B$?
(d) If $(a, b) \in A \times B$, what is the inverse $(a, b)^{-1}$ equal to?
(e) Assume that neither of A and B is the trivial group. Prove that these four subgroups are normal in $A \times B$:

$$\{e_A\} \times \{e_B\}, \qquad A \times \{e_B\}, \qquad \{e_A\} \times B, \qquad A \times B$$

14. (a) True or false? The order of the group D_n is the same as the order of the group $\mathbb{Z}_2 \times \mathbb{Z}_n$.
(b) True or false? The group D_n is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_n$.
(c) True or false? The group $\mathbb{Z}_{14}$ is isomorphic to the group $\mathbb{Z}_2 \times \mathbb{Z}_7$.
(d) True or false? The group $\mathbb{Z}_{16}$ is isomorphic to the group $\mathbb{Z}_4 \times \mathbb{Z}_4$.
(e) Is $\mathbb{Z}_{12}$ isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_6$?
(f) Which direct product is isomorphic to $\mathbb{Z}_{12}$?

15. Let H be a subgroup of G.

    (a) What does the notation G/H mean?

    (b) When is G/H a group?

    (c) If G/N is a quotient group, what is the binary operation of the quotient group G/N?

    (d) Consider the symmetric group S_3 and a subgroup $H := \langle(1\ 2)\rangle$. Is the set $S_3/\langle(1\ 2)\rangle$ a quotient group? Prove your answer. If it is a quotient group, what is it isomorphic to?

    (e) Consider the symmetric group S_3 and a subgroup $J := \langle(1\ 2\ 3)\rangle$. Is S_3/J a quotient group? Prove your answer. If it is a quotient group, what is it isomorphic to?

16. The following are all normal subgroups of D_4:

    (a) The trivial subgroup $\{e\}$,

    (b) the only normal subgroup of order 2, $\langle r^2\rangle$,

    (c) all the subgroups of order 4: $\langle r\rangle$, $\langle r^2, f\rangle$, $\langle r^2, rf\rangle$, and

    (d) D_4 itself.

    For each N above, what familiar group is D_4/N isomorphic to?

17. Let H be a subgroup of G, and consider the subset of G denoted by

$$\mathrm{Nor}_G(H) = \{g \in G : gH = Hg\} = \{g \in G : gHg^{-1} = H\}.$$

    Note: this set $\mathrm{Nor}_G(H)$ is often called the *normalizer of H in G*; it is the set of elements in G that "vote" in favor of H's normality.

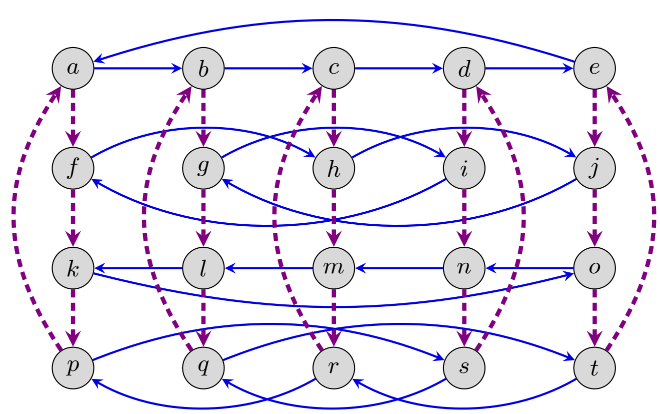
    (a) Prove that $\mathrm{Nor}_G(H)$ is a subgroup.

    (b) What is the smallest that $\mathrm{Nor}_G(H)$ can be?

    (c) What is the largest $\mathrm{Nor}_G(H)$ can be?

    (d) When does the latter happens?

18. Let G be the group whose Cayley diagram is shown below, and suppose e is the identity element. Consider the subgroups $A = \langle a\rangle = \{a, b, c, d, e\}$ and $J = \langle j\rangle = \{e, j, o, t\}$.



    Carry out the following steps for both of the subgroups A and J. List the cosets element-wise.

(a) Write G as a disjoint union of the left cosets of A. Write G as a disjoint union of the left cosets of J.

(b) Write G as a disjoint union of the right cosets of A. Write G as a disjoint union of the right cosets of J.

(c) Use your coset computation to immediately compute the normalizer of the subgroup. Based on the computation for the normalizer, what you can say about this subgroup?

(d) If G/A is a group, perform the quotient process and draw the resulting Cayley diagram for G/A.
If G/J is a group, perform the quotient process and draw the resulting Cayley diagram for G/J.

19. The *center* of a group G is the set

$$Z(G) = \{z \in G \mid gz = zg, \text{ for all } g \in G\} = \{z \in G \mid gzg^{-1} = z, \text{ for all } g \in G\}.$$

It is a subgroup of G.

a. Prove that $Z(G)$ is normal in G by showing $ghg^{-1} \in H$ for all $h \in H, g \in G$ ("closed under conjugation").

b. Compute the center of $\mathbb{Z}_6$.

c. Compute the center of D_4.

d. Compute the center of D_5.

e. Consider the group A_3 of even permutations. Compute the center of A_3.

f. Consider the group A_n of even permutations, where $n \geq 4$. Prove that $(1\ 2\ 3)$ is not in the center of A_n by producing another even permutation which does not commute with $(1\ 2\ 3)$.

g. Let $n \geq 4$. Prove that $(1\ 2)(3\ 4)$ is not in the center of A_n.

h. Compute the center of A_4
Hint: A non-identity permutation in S_4 is an even permutation if and only of its cycle notation is of the form $(ab)(cd)$ or (abc). (Make sure you can prove this!)
Do $(ab)(cd)$ and (abc) commute?

i. Compute the center of S_4.
Hint: Every non-identity permutation in S_4 can be written in the form (ab), (abc), $(abcd)$, and $(ab)(cd)$. Can you find a permutation that does not commute with (ab)? With $(abcd)$?

j. Compute the center of S_2.

k. Prove that "the center of a direct product is the direct product of the centers", that is, $Z(A \times B) = Z(A) \times Z(B)$.

20. Notation/Definition: Let G be a group and $x \in G$.

- The *conjugacy* class of x is the set $\text{cl}_G(x) := \{gxg^{-1} \mid g \in G\}$.
- Let $Z(G)$ be the set $\{z \in G \mid gz = zg \text{ for all } g \in G\}$.

(a) Prove that $\text{cl}_G(x) = \{x\}$ if and only if $x \in Z(G)$.

(b) Suppose N is a normal subgroup of G. Prove that if $x \in N$, then $\text{cl}_G(x) \subset N$.

21. You can use the following fact.

Proposition 1. For any $\sigma \in S_n$, we have $\sigma\ (a_1\ a_2\ \ldots\ a_k)\ \sigma^{-1} = (\,\sigma(a_1)\ \sigma(a_2)\ \ldots\ \sigma(a_k)\,)$.

(a) Let x be a k-cycle. Prove that $y \in S_n$ is conjugate to x iff y is a k-cycle.

(b) Prove that (12) and (14) in S_6 are conjugate by finding a permutation $p \in S_6$ such that $p^{-1}(12)p = (14)$.

(c) List all permutations in S_4 which are conjugate to (1234). Use the fact from part (a).

Proposition 2. Let $f : G_1 \to G_2$ be a homomorphism of groups. Then

(a) If e_1 is the identity of G_1, then $f(e_1)$ is the identity of G_2.

(b) For any element $g \in G_1$, $f(g^{-1}) = [f(g)]^{-1}$.

(c) If H_1 is a subgroup of G_1, then $f(H_1)$ is a subgroup of G_2.

(d) (i) If H_2 is a subgroup of G_2, then $f^{-1}(H_2) = \{g \in G_1 : f(g) \in H_2\}$ is a subgroup of G_1.
(ii) Furthermore, if H_2 is normal in G_2, then $f^{-1}(H_2)$ is normal in G_1.

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22. Prove all parts of Proposition 2.

23. (a) Let $f : G_1 \to G_2$ be a homomorphism of groups. Prove that the kernel of f is a normal subgroup of G_1.
    (b) Let $f : G \to H$ be a group homomorphism. Show that if $\ker(f)$ is the trivial group $\{1_G\}$ then f is injective.

24. (a) Let $f : G_1 \to G_2$ be a *surjective* homomorphism. Prove that, if $N \lhd G_1$, then $f(N)$ is normal in G_2.
    (b) If $f : G_1 \to G_2$ is a homomorphism and N is a normal subgroup of G_1, is it possible that $f(N)$ is not normal in G_2? If so, give an example.

25. Let $\phi : (\mathbb{Z}, +) \to (\mathbb{Z}, +)$ be the map given by $\phi(n) = 7n$ for $n \in \mathbb{Z}$. Find the kernel and the image of ϕ.

26. Consider the group homomorphism $f : (\mathbb{R}, +) \to (\mathbb{C}^*, \times)$ defined by

$$f(\theta) = \cos\theta + i\sin\theta.$$

    (a) Find the kernel of f and the image of f.
    (b) Give an isomorphism (bijective group homomorphism) from the kernel of f to $(\mathbb{Z}, +)$.

27. Let G be a group and let g be some element in G. Consider the group homomorphism $f : \mathbb{Z} \to G$ given by

$$f(n) = g^n.$$

    (a) If the order of g is infinite, what is the kernel of f? Justify.
    (b) If the order of g is finite, say m, what is the kernel of f? Justify.

28. True or false? Given two groups A and B, there exists a homomorphism from A to B. Prove your answer.

29. Given a homomorphism $f : G \to H$ define a relation $\sim$ on G by $a \sim b$ if $f(a) = f(b)$ for $a, b \in G$.
    (a) Show that this relation is an equivalence relation.
    (b) Describe the equivalence classes. How many classes are there?