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Pattern-avoiding Motzkin tableaux from box-ball systems

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Abstract. A box-ball system (BBS) is a dynamical system consisting of an infinite strip of boxes, each of which is either empty or contains a ball labeled by a unique positive integer. Beginning with some permutation on $[n]$, we place the permutation in consecutive boxes in the strip; and at each time step, we move each ball to the next empty box to its right, starting with the ball labeled 1 and ending with n. Eventually, the balls rearrange themselves into increasing blocks that continue to move together, called solitons. The tableau that we get by stacking these solitons is called the soliton decomposition of the BBS. We call a permutation "good" if its soliton decomposition is a standard Young tableau. The goodness of a permutation is determined by its Robinson-Schensted (RS) recording tableau, and so we call a standard tableau T "good" if T is the recording tableau of a good permutation. In this project, we characterize good tableaux T using pattern avoidance of T and the inverse of the column reading word of T. We use this characterization to prove that the good tableaux of size n are counted by the n-th Motzkin number. Along the way, we also show that the good permutations are closed under consecutive pattern containment.

Keywords: Box-ball systems, Motzkin numbers, RSK tableaux, Pattern avoidance

1 Introduction

The n-th *Motzkin number* m_n is the number of ways to draw nonintersecting chords between n labeled points on a circle [9]. They count a variety of other objects (see [10, A001006]), called Motzkin objects. The paper [2] describes fourteen of these objects and

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also explains how the Motzkin numbers are closely related to Catalan numbers. The Motzkin numbers can be defined by the two-term recurrence relation

$$m_n = m_{n-1} + \sum_{k=0}^{n-2} m_k \, m_{n-k-2} \tag{1.1}$$

with $m_0 = 1$, $m_1 = 1$. The first few Motzkin numbers, for $n = 0, 1, ..., 13$, are

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835.$$

1.1 Box-ball systems and their soliton decompositions

A *box-ball system* (BBS) is a dynamical system consisting of discrete time states. At each time state, we have a BBS configuration: finitely many labeled balls in an infinite strip of boxes, such that each box is either empty or contains one ball.

Let S_n denote the set of permutations on $[n] = \{1, 2, \ldots, n\}$. At time $t = 0$, for some $w \in S_n$, we have a BBS configuration given by the balls $1, 2, \ldots, n$ placed in n consecutive boxes following the one-line notation of w. One box-ball move is the process of letting each ball jump to the nearest empty box to its right, beginning with the ball 1 and ending with the ball n. Given a BBS configuration at time t, we reach the BBS configuration at time $t + 1$ by applying one box-ball move.

For example, begin with $w = 452361 \in S_6$. Then, we perform a box-ball move by first moving 1 to the first open box on its right, then 2, and so on until we move 6:

$t=0$	...		4	5	2	3	6	1					...
	...		4	5	2	3	6		1				...
	...		4	5		3	6	2	1				...
	...		4	5			6	2	1	3			...
	...			5	4		6	2	1	3			...
	...				4	5	6	2	1	3			...
$t=1$	...				4	5		2	1	3	6		...

We can then observe what happens to the permutation after several box-ball moves:

$t=0$	...		4	5	2	3	6	1																	...

$t = 1$	...				**4**	**5**		**2**	**1**	**3**	**6**														...
$t = 2$	...						**4**	**5**	**2**			**1**	**3**	**6**											...
$t = 3$	...								**4**	**2**	**5**				**1**	**3**	**6**								...
$t = 4$	...									**4**		**2**	**5**					**1**	**3**	**6**					...
$t = 5$	...										**4**			**2**	**5**						**1**	**3**	**6**		...

In this example, in each box-ball move after $t = 3$, the increasing sequence **136** travels three spaces to the right, the increasing pair **25** travels two spaces to the right, and the singleton **4** travels one space to the right.

These increasing sequences are called *solitons* — maximal consecutive increasing sequences of balls that are preserved by all future box-ball moves. After a finite number of box-ball moves, every box-ball system will reach a steady state, decomposing into solitons whose sizes are weakly increasing from left to right, that is, forming an integer partition of n.

In this paper, we define a tableau to be an array of positive integers whose row sizes are weakly decreasing (that is, the shape of a tableau is an integer partition). The *soliton decomposition* of a box-ball system, called SD for short, is the tableau where the first row is the rightmost soliton, the second row is the second-rightmost soliton, and so on. Note that each row of this tableau is necessarily an increasing sequence, but the columns are not necessarily increasing. The shape of SD is called the *BBS soliton partition*.

Given a permutation w, its soliton decomposition $\mathrm{SD}(w)$ is defined to be the soliton decomposition of the box-ball system containing the configuration w. The permutation $w = \mathbf{452361}$ in our example has soliton decomposition

$$\mathrm{SD}(w) = \begin{array}{|c|c|c|}\hline \mathbf{1} & \mathbf{3} & \mathbf{6} \\ \hline \mathbf{2} & \mathbf{5} & \multicolumn{1}{c}{} \\ \cline{1-2} \mathbf{4} & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

Our version of the box-ball system, known as the *multicolor box-ball system*, was introduced in [12]. For more details, see the survey [7].

1.2 RS Correspondence and good tableaux

A tableau is called *standard* if its entries are the integers in $[n]$, each appearing exactly once, and if each row and each column is increasing. We will use a tool called the *Robinson–Schensted (RS) insertion algorithm* to study the box-ball system. It is a well-known bijection

$$w \mapsto (\mathrm{P}(w), \mathrm{Q}(w))$$

from S_n onto pairs of standard size-n tableaux of the same shape [11]. The shape of $\mathrm{P}(w)$ is called the *RS partition* of w. In this paper, upright P, Q are used as functions, whereas italicized P, Q denote specific tableaux. The tableau $\mathrm{P}(w)$ is called the *insertion tableau* of w, and the tableau $\mathrm{Q}(w)$ is called the *recording tableau* of w. For details on this algorithm, see for example the textbook [5, Section 4.1].

For example, if $w = 452361$ as in our running example, then

$$w = 452361 \quad \overset{\mathrm{RS}}{\mapsto} \quad \mathrm{P}(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}, \quad \mathrm{Q}(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$$

Observe that in this example $\mathrm{P}(w) = \mathrm{SD}(w)$. This property holds only for a special class of permutations that are the focus of this paper.

Definition 1.1. A permutation w is called *good* if $\mathrm{P}(w) = \mathrm{SD}(w)$ and *bad* otherwise.

In the 1970s, Greene showed that the RS partition of a permutation and its conjugate record the numbers of disjoint unions of increasing and decreasing sequences of the permutation([6, Theorem 3.1]). Lewis, Lyu, Pylyavskyy, and Sen recently showed that the BBS soliton partition of a permutation and its conjugate record a localized version of Greene's theorem statistics [8, Lemma 3.5]; for a survey, see [3, Section 2.2]. The paper [3], which heavily used Greene's theorem and localized Greene's theorem statistics, studied properties of good vs. bad permutations in connection to their insertion tableaux. The following allows us to determine the goodness of permutations from the shape of their soliton decompositions.

Theorem 1.2 ([3, Theorem 4.2])**.** Let w be a permutation. Then $\mathrm{SD}(w) = \mathrm{P}(w)$ iff $\mathrm{SD}(w)$ is a standard tableau iff $\mathrm{sh}\,\mathrm{SD}(w) = \mathrm{sh}\,\mathrm{P}(w)$.

A subsequent paper [1] showed that the recording tableau of a permutation determines its goodness.

Theorem 1.3 ([1, Theorem A])**.** If $v, w \in S_n$ with $\mathrm{Q}(v) = \mathrm{Q}(w)$, then we have:

1. $\mathrm{sh}\,\mathrm{SD}(v) = \mathrm{sh}\,\mathrm{SD}(w)$
2. v is good iff w is good

In view of this result, it is natural for the authors of [1] to define the following.

Definition 1.4. A standard tableau T is *good* if $\mathrm{Q}(w) = T$ for some good permutation w.

They also gave two conjectures, Conjectures 1.5 and 1.6.

Conjecture 1.5 ([1, Conjecture 8.5])**.** If a permutation w is good, then the standardization of every consecutive subpattern of w is also good.

As a corollary to the machinery we constructed in Section 2, we are able to prove that removing the largest entry from a good tableau results in another good tableau (Proposition 2.15); this shows that good permutations are preserved under taking prefixes. It was shown in [4] that good permutations are also preserved under taking suffixes. This concludes the proof of Conjecture 1.5.

Conjecture 1.6 ([1, Conjecture 8.6])**.** The number of size-n good tableaux is equal to the nth Motzkin number.

The main goal of the present paper is to sketch a proof of Conjecture 1.6. The rest of this paper is organized as follows. We characterize good tableaux using pattern-avoidance and a concept called φ-increasing in Section 2. In Section 3.1 we describe three goodness-preserving operations called tilde multiplication, column bump, and row wrap, which were inspired by [5, Section 1.1]. In Section 3.2, we recursively construct a class of good tableaux GT_n whose cardinality matches the two-term recurrence relation (1.1). We prove in Theorem 3.12 that the set of all good tableaux of size n is equal to GT_n, and the cardinality of GT_n is given in Corollary 3.13.

2 Characterization of good tableaux

Given a tableau T of size n, let $\operatorname{sh} T = \lambda^T := (\lambda_1^T, \lambda_2^T, \dots)$, called the *shape* of T, be the partition of n such that λ_i^T is the width of the i-th row of T. Let $\mu^T = (\mu_1^T, \mu_2^T, \dots)$ denote the conjugate of λ^T. Thus, μ_i^T denotes the height of the i-th column of T.

2.1 Column superstandard words

In this section we discuss the notion of a column superstandard word associated to a tableau; we will use it in the next section to characterize good tableaux.

Definition 2.1. A tableau of size n is called *column superstandard* if, when reading its columns from top to bottom and left to right, we get the integers $1, 2, 3, \dots, n$ in this order.

Given a standard tableau Q, let $\mathrm{CSS}(Q)$ denote the unique column superstandard tableau of shape $\operatorname{sh} Q$. Let col_i^Q denote the entries of the i-th column of $\mathrm{CSS}(Q)$, written as a decreasing sequence. Note that $\mu_i^Q = \mathsf{len}(\mathrm{col}_i^Q)$. We will usually write col_i^Q as col_i when Q is understood.

Given a standard tableau Q, we refer to the permutation $\pi = \mathrm{RS}^{-1}(\mathrm{CSS}(Q), Q)$ as the *column superstandard word* of Q.

Example 2.2. Consider a standard tableau Q and the corresponding $\mathrm{CSS}(Q)$

$$\mathrm{CSS}(Q) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 8 & 14 & 19 & 21 & 22 \\ \hline 2 & 9 & 15 & 20 \\ \cline{1-4} 3 & 10 & 16 \\ \cline{1-3} 4 & 11 & 17 \\ \cline{1-3} 5 & 12 & 18 \\ \cline{1-3} 6 & 13 \\ \cline{1-2} 7 \\ \cline{1-1} \end{array} \qquad Q = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 6 & 7 & 11 \\ \hline 4 & 5 & 10 & 22 \\ \cline{1-4} 8 & 9 & 14 \\ \cline{1-3} 12 & 13 & 17 \\ \cline{1-3} 15 & 16 & 21 \\ \cline{1-3} 18 & 20 \\ \cline{1-2} 19 \\ \cline{1-1} \end{array}$$

Then $\mathrm{col}_1 = \mathbf{7654321}$, $\mathrm{col}_2 = \mathbf{13\ 12\ 11\ 10\ 9\ 8}$, $\mathrm{col}_3 = \mathbf{18\ 17\ 16\ 15\ 14}$, $\mathrm{col}_4 = \mathbf{20\ 19}$, $\mathrm{col}_5 = \mathbf{21}$, $\mathrm{col}_6 = \mathbf{22}$, and the column superstandard word of Q is

$$\begin{aligned} \pi &= \mathrm{RS}^{-1}(\mathrm{CSS}(Q), Q) \\ &= \mathbf{7\ 13\ 18\ 6\ 12\ 20\ 21\ 5\ 11\ 17\ 22\ 4\ 10\ 16\ 3\ 9\ 15\ 2\ 1\ 8\ 14\ 19} \end{aligned}$$

Observe that each col_i occurs as a subsequence of π, as we state below in Lemma 2.3.

Lemma 2.3. Let Q be a standard tableau and π be its column superstandard word. Then, for each i, the (decreasing) sequence col_i is a subsequence of π.

In light of this lemma, we can view col_i as decreasing subsequences of the column superstandard word of a given Q.

Furthermore, the positions in which the elements of col_i appear in π are related to the column of the corresponding entry in Q. In Example 2.2, the column superstandard word $\pi = \mathrm{RS}^{-1}(\mathrm{CSS}(Q), Q)$ of Q is equal to

$$\begin{pmatrix} \pi(1) & \pi(2) & \pi(3) & \pi(4) & \pi(5) & \pi(6) & \pi(7) & \pi(8) & \pi(9) & \pi(10) & \pi(11) & \pi(12) & \pi(13) & \pi(14) & \pi(15) & \pi(16) & \pi(17) & \pi(18) & \pi(19) & \pi(20) & \pi(21) & \pi(22) \\ \mathbf{7} & \mathbf{13} & \mathbf{18} & \mathbf{6} & \mathbf{12} & \mathbf{20} & \mathbf{21} & \mathbf{5} & \mathbf{11} & \mathbf{17} & \mathbf{22} & \mathbf{4} & \mathbf{10} & \mathbf{16} & \mathbf{3} & \mathbf{9} & \mathbf{15} & \mathbf{2} & \mathbf{1} & \mathbf{8} & \mathbf{14} & \mathbf{19} \end{pmatrix}$$

Observe that the positions in which elements of col_1 appear in π are $\mathbf{1, 4, 8, 12, 15, 18, 19}$ which are exactly the entries of column 1 of Q. Similarly, the positions in which elements of col_2 appear in π are $\mathbf{2, 5, 9, 13, 16, 20}$, which are the entries of column 2 of Q. This relationship is established in Lemma 2.6.

Definition 2.4. The *column reading word* of a tableau T, denoted $\mathrm{crw}(T)$, is the word obtained by reading the columns of T, read from bottom to top, from left to right.

It is well-known (see [5, Section 2.3]) that $\mathrm{P}(\mathrm{crw}(T)) = T$ for any standard tableau T. In Example 2.2, the column reading word for the tableau Q is

$$\mathrm{crw}(Q) = \mathbf{19\ 18\ 15\ 12\ 8\ 4\ 1}\ \ \mathbf{20\ 16\ 13\ 9\ 5\ 2}\ \ \mathbf{21\ 17\ 14\ 10\ 3}\ \ \mathbf{22\ 6}\ \ \mathbf{7}\ \ \mathbf{11},$$

and $\mathrm{RS}(\mathrm{crw}(Q)) = (Q, \mathrm{CSS}(Q))$.

Lemma 2.5. The column superstandard word of T is the inverse of the column reading word $\mathrm{crw}(T)$ of T.

Lemma 2.6. Let π be the column superstandard word of a standard tableau Q. Let x, y be such that $\pi(x) = y$. Then $Q(r, c) = x$ iff $\mathrm{col}_c(r) = y$, i.e., we have

$$\mathrm{col}_c(r) = \pi(Q(r, c))$$

2.2 Characterizing goodness via column superstandard words

In this section we characterize good tableaux by analyzing partitions of particular subwords of the column superstandard word of a tableau.

Definition 2.7. Given a standard tableau Q, let π be its column superstandard word. Viewing $\mathrm{col}_1^Q, \mathrm{col}_2^Q, \ldots$ as decreasing subsequences of π (see Lemma 2.3), for $1 \le i \le$ |columns of Q|, let

$$u_i^Q = \mathrm{col}_i^Q \sqcup \mathrm{col}_{i+1}^Q \sqcup \ldots,$$

that is, we let u_i^Q be the subsequence of π consisting of the letters in $\mathrm{col}_i^Q \sqcup \mathrm{col}_{i+1}^Q \sqcup \ldots$.

Lemma 2.8. Let Q be a standard tableau and $1 \le i \le$ |columns of Q|. Then $u_i^Q(1) = \mathrm{col}_i^Q(1)$.

In view of Lemma 2.8, we can define the following.

Definition 2.9. Let Q be a standard tableau and $1 \le i \le$ |columns of Q|. Let $u_i = u_i^Q$, $\mathrm{col}_i = \mathrm{col}_i^Q$, and $\mu_i = \mu_i^Q$. Define $\varphi_1^{u_i}, \varphi_2^{u_i}, \ldots, \varphi_{\mu_i}^{u_i}$ to be the consecutive subsequences of u_i starting with $\mathrm{col}_i(1), \mathrm{col}_i(2), \ldots, \mathrm{col}_i(\mu_i)$, respectively, such that

$$u_i = \varphi_1^{u_i} \cdot \varphi_2^{u_i} \cdot \ldots \cdot \varphi_{\mu_i}^{u_i}$$

We say Q is *φ-increasing* if for each $1 \le i \le \lambda_1^Q$ and each $1 \le j \le \mu_i^Q$, $\varphi_j^{u_i}$ is increasing.

Example 2.10. Consider π from Example 2.2. For $i = 1$,

$$\pi = u_1 = \underbrace{\boxed{7\ 13\ 18}}_{\varphi_1^{u_1}}\ \underbrace{\boxed{6\ 12\ 20\ 21}}_{\varphi_2^{u_1}}\ \underbrace{\boxed{5\ 11\ 17\ 22}}_{\varphi_3^{u_1}}\ \underbrace{\boxed{4\ 10\ 16}}_{\varphi_4^{u_1}}\ \underbrace{\boxed{3\ 9\ 15}}_{\varphi_5^{u_1}}\ \underbrace{\boxed{2}}_{\varphi_6^{u_1}}\ \underbrace{\boxed{1\ 8\ 14\ 19}}_{\varphi_7^{u_1}}$$

For $i = 2$ and 3, we have $u_2 = \mathrm{col}_2 \sqcup \mathrm{col}_3 \ldots$ and $u_3 = \mathrm{col}_3 \sqcup \mathrm{col}_4 \ldots$ as follows.

$$u_2 = \underbrace{\boxed{13\ 18}}_{\varphi_1^{u_2}}\ \underbrace{\boxed{12\ 20\ 21}}_{\varphi_2^{u_2}}\ \underbrace{\boxed{11\ 17\ 22}}_{\varphi_3^{u_2}}\ \underbrace{\boxed{10\ 16}}_{\varphi_4^{u_2}}\ \underbrace{\boxed{9\ 15}}_{\varphi_5^{u_2}}\ \underbrace{\boxed{8\ 14\ 19}}_{\varphi_6^{u_2}}$$

$$u_3 = \underbrace{\boxed{18\ 20\ 21}}_{\varphi_1^{u_3}}\ \underbrace{\boxed{17\ 22}}_{\varphi_2^{u_3}}\ \underbrace{\boxed{16}}_{\varphi_3^{u_3}}\ \underbrace{\boxed{15}}_{\varphi_4^{u_3}}\ \underbrace{\boxed{14\ 19}}_{\varphi_5^{u_3}}$$

and one can compute u_4, u_5 and u_6 in a similar fashion.

Proposition 2.11. If Q is good, then Q is φ-increasing.

2.3 Classifying good tableaux via pattern avoidance

Definition 2.12 (Tableaux patterns)**.** For a standard tableau Q we call an ordered collection $(j_x, i_x), (j_y, i_y), (j_z, i_z)$ of positions in Q an *abc pattern* if they are such that

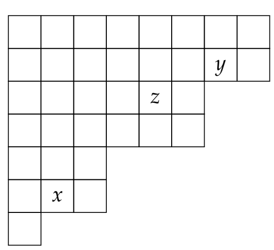
$$j_x = \mu^Q_{i_x},\ \ i_x < i_z \leq i_y, \text{ and } Q(j_x, i_x) < Q(j_y, i_y) < Q(j_z, i_z)$$

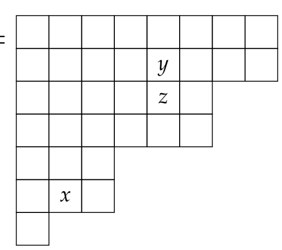
For a standard tableau Q we call an ordered collection $(j_x, i_x), (j_y, i_y), (j_z, i_z)$ of positions in Q an *abcd pattern* if they are such that

$$j_x < \mu^Q_{i_x},\ \ i_x < i_z \leq i_y, \text{ and } Q(j_x, i_x) < Q(j_y, i_y) < Q(j_z, i_z) < Q(j_x + 1, i_x)$$

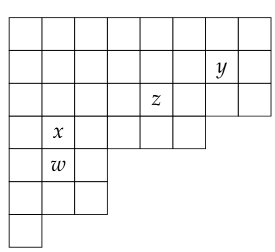
We say that Q is *abc-avoiding* (resp. *abcd-avoiding*) if Q has no *abc* (resp. *abcd*) pattern.

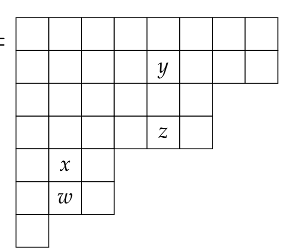
Remark 2.13. Depicted below are two examples of Q tableaux with an *abc* pattern, letting $x = Q(j_x, i_x), y = Q(j_y, i_y)$, and $z = Q(j_z, i_z)$. Note that the condition $j_x = \mu^Q_{i_x}$ means that x appears in a south-most position.

$Q =$ 

or $Q =$ 

In the next two tableaux, Q exhibits an *abcd* pattern. We write $x = Q(j_x, i_x)$, $y = Q(j_y, i_y)$, $z = Q(j_z, i_z)$, and $w = Q(j_x + 1, i_x)$.

$Q =$ 

or $Q =$ 

Theorem 2.14 (Characterization of goodness)**.** Let Q be a standard tableau. Then the following are equivalent.

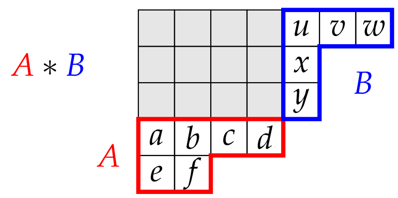


Figure 1: Illustration for an alternative description of the product, using jeu de taquin

1. Q is good.
2. Q is φ-increasing.
3. Q has no abc or $abcd$ patterns.

Proof (sketch). That (1) implies (2) is Proposition 2.11. The rest of the proof use pattern-avoidance condition of Definition 2.12 and localized Greene's statistics. □

We can use Theorem 2.14 to help us prove Proposition 2.15, which is useful in the inductive proof of the main theorem (Theorem 3.12) and also interesting for studying consecutive permutation patterns.

Proposition 2.15. If Q is a good tableau of size n, then $Q \setminus \{n\}$ is also a good tableau.

3 Proof sketch that good tableaux are Motzkin objects

3.1 Goodness-preserving operations on tableaux

The goal of this section is to define a goodness-preserving operation on tableaux for every operation in the Motzkin recursion: column bump, row wrap, and tilde multiplication.

Definition 3.1 (Tilde product)**.** The tilde product $T_1 \widetilde{\times} T_2$ can be constructed as follows:

1. Start with the standard tableau T_2, and let n_2 denote the size of T_2.
2. Let $\overline{T_1}$ denote the result of replacing every entry j in T_1 with $j + n_2$. For each i, append the i-th column of $\overline{T_1}$ to (below) the i-th column of T_2.

Remark 3.2. We chose the notation $\widetilde{\times}$ because $T_1 \widetilde{\times} T_2 = \overline{T_1} \times T_2$, where $\times$ is an associative binary operation on the set of semistandard tableaux; this product can be constructed using a process called *jeu de taquin*. Given two tableaux A and B, form a skew tableau denoted $A * B$ by taking a rectangle of empty squares with the same number of columns as A and the same number of rows as B and putting A below and B to the right of this rectangle; see Figure 1. Then $A \times B$ is equal to the *rectification* of $A * B$, a tableau obtained by performing a series of slides; see [5], Section 1.1 (page 11) and Section 1.2 (Claim 3). In our case, since every entry in T_2 is smaller than every entry in $\overline{T_1}$, the rectification of $\overline{T_1} * T_2$ can be achieved by sliding all cells of T_2 to the left and each column of $\overline{T_1}$ up.

Definition 3.3. Let $\underline{T}$ denote the result of replacing every entry j in T with $j-(m-1)$, where m is the minimum entry in T. Given a skew tableau S, let $\operatorname{flushup}(S)$ denote the result of successively sliding each column of S upwards to be flush with the top row.

Example 3.4. If $T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$, $T_2 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$, then $\overline{T_1} = \begin{array}{|c|c|c|}\hline 3 & 4 & 7 \\ \hline 5 & 6 \\ \cline{1-2}\end{array}$ and $T_1 \mathbin{\tilde{\times}} T_2 = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$.

If $T = \begin{array}{|c|c|c|}\hline 4 & 5 & 8 \\ \hline 6 & 7 \\ \cline{1-2}\end{array}$, then we reduce all entries of T by 3 to construct $\underline{T} = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$.

If $S = \begin{array}{ccc}\cline{3-3} & & \multicolumn{1}{|c|}{3} \\ \cline{2-3} & \multicolumn{1}{|c|}{4} & \\ \cline{1-2} \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{5} & \\ \cline{1-2} \multicolumn{1}{|c|}{2} & & \\ \cline{1-1}\end{array}$, then $\operatorname{flushup}(S) = \begin{array}{|c|c|c|}\hline 1 & 4 & 3 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$.

We now introduce two ways of obtaining a new standard tableau from a standard tableau, called column bump and row wrap. Applying these operations to a good tableau will result in good tableaux, as stated in Proposition 3.9.

Definition 3.5 (Column bump)**.** Let T be a standard tableau of size n. We construct the *column bump* of T, denoted $\operatorname{bump}(T)$, which is a new tableau of size $n+1$, as follows:

- Let T' be the result of increasing every entry in the first column of T by 1; and fixing the entries in all other columns of T
- Prepend 1 to the top of the first column of T'.

We say T' is *column bumped* if T' is the column bump of a standard tableau.

Definition 3.6 (Row wrap)**.** Let T be a standard tableau of size n. We construct the *row wrap* of T, denoted $\operatorname{wrap}(T)$, which is a new tableau of size $n+2$, as follows:

- Let T' be the result of increasing every entry of T by 1;
- Prepend 1 to the beginning and append $n+2$ to the end of the first row of T'.

A standard tableau T' is called *wrapped* if T' is the row wrap of a standard tableau.

Example 3.7. If $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}$, then $\operatorname{bump}(T) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 7 \\ \cline{1-2} 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}$, $\operatorname{wrap}(T) = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 8 \\ \hline 5 & 7 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$.

In each of $\operatorname{wrap}(T)$ and $\operatorname{bump}(T)$, an "edited copy" of the tableau T is enclosed in bold.

Proposition 3.8. Suppose A_1, A_2, B_1, B_2 are standard tableaux such that $A_1 \widetilde{\times} \operatorname{wrap}(B_1) = A_2 \widetilde{\times} \operatorname{wrap}(B_2)$. Then $A_1 = A_2$, $B_1 = B_2$.

The following can be shown using Theorem 2.14.

Proposition 3.9. Suppose Q, T_1, T_2 are standard tableaux. Then we have the following.

1. Q is good iff $\operatorname{bump}(Q)$ is good iff $\operatorname{wrap}(Q)$ is good.
2. $T_1 \widetilde{\times} T_2$ is good iff T_1 and T_2 are good.

3.2 Recursive construction of a class of good tableaux

Proposition 3.9 tells us that the operations of column bumping, wrapping, and taking tilde products of good tableaux all preserve goodness. We use these operations to recursively define a class of good tableaux that are counted by the Motzkin numbers.

Definition 3.10. Let $\mathrm{GT}_0 = \{\varnothing\}$ where $\varnothing$ denotes the empty tableau, and let $\mathrm{GT}_1 = \{\boxed{1}\}$. Then, for $n \geq 2$,

1. for each $Q \in \mathrm{GT}_{n-1}$, let
$$\operatorname{bump}(Q) \in \mathrm{GT}_n$$
2. for $0 \leq k \leq n-2$, for each pair of $Q_1 \in \mathrm{GT}_k$ and $Q_2 \in \mathrm{GT}_{n-k-2}$, let
$$Q_1 \widetilde{\times} \operatorname{wrap}(Q_2) \in \mathrm{GT}_n$$

The following lemma will be helpful in the proof of the main theorem.

Lemma 3.11. Let Q be a good tableau of size n and let Q_{n-1} denote $Q \setminus \{n\}$.

1. If Q_{n-1} is column bumped, then Q is also column bumped.
2. If Q_{n-1} is wrapped, then either $Q = \boxed{1} \widetilde{\times} Q_{n-1}$ or Q is wrapped.
3. If $Q_{n-1} = A \widetilde{\times} B$ where B is wrapped and A is a nonempty standard tableau, then Q is either wrapped or $Q = \underline{\operatorname{flushup}(Q \setminus B)} \widetilde{\times} B$.

Theorem 3.12 (Main Theorem)**.** For $n \in \mathbb{Z}_{\geq 0}$, GT_n is equal to the good tableaux of size n.

Proof (sketch). Induct on n, and apply Proposition 2.15, Proposition 3.9, and Lemma 3.11. □

The good tableaux of size n are counted by the n-th Motzkin number.

Corollary 3.13. Let $n \geq 1$. Then $|\mathrm{GT}_n| = m_n$, the n^{th} Motzkin number.

Bad involutions are counted by [10, A000085], the sequence counting involutions, minus the Motzkin numbers.

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