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Heap Birkhoff subpolytopes

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Abstract. Given the heap H of an arbitrary reduced word in a Coxeter group of type A, we define a Birkhoff subpolytope $\mathsf{Birk}(H)$ whose vertices are permutation matrices corresponding to the order ideals of H. We compare $\mathsf{Birk}(H)$ with $\mathcal{O}(H)$, the $|H|$-dimensional order polytope of H whose vertices are indicator vectors of the order ideals of H. For the heaps H such that $\mathsf{Birk}(H)$ and $\mathcal{O}(H)$ have the same dimension, we describe an integral equivalence from $\mathsf{Birk}(H)$ to $\mathcal{O}(H)$. We classify these heaps using a novel combinatorial model, a collection of digraphs denoted $\Gamma_i(H)$: $\mathsf{Birk}(H)$ is integrally equivalent to $\mathcal{O}(H)$ if and only if $\Gamma_i(H)$ is a forest for all i. We then define a Birkhoff polytope for a type B heap and prove an analogous result in this setting.

Keywords: Birkhoff polytope, order polytope, heap, reduced word, Coxeter group

1 Introduction

Given a finite poset H, the *order polytope* of H, denoted $\mathcal{O}(H)$, is a well-studied convex polytope whose vertices are the indicator vectors of the order ideals of H. It is a full-dimensional polytope living in $\mathbb{R}^{|H|}$ whose normalized volume is the number of linear extensions of H [7].

The *Birkhoff polytope* is defined as the convex polytope of $m \times m$ doubly stochastic matrices, i.e. matrices with non-negative entries whose rows and columns all sum to 1. Alternatively it can be defined as the convex hull of permutation matrices $X(w)$ for all w in $S_m = A_{m-1}$ [3].

In [5], Davis and Sagan studied the convex hull of 132 and 312 avoiding permutation matrices, a subpolytope of the Birkhoff polytope. They proved that the normalized

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volume of this polytope is the number of longest chains in the type A_{m-1} Tamari lattice (OEIS A003121). Since 132 and 312 avoiding permutations are exactly the c-singletons for the Coxeter element $c = (12\dots m)$, we defined, for any Coxeter element c, a (type A) Birkhoff subpolytope $\mathsf{Birk}(c)$ to be the convex hull of permutation matrices $X(w)$ for all c-singletons w in A_{m-1} [1]. We then proved that $\mathsf{Birk}(c)$ is integrally equivalent to the order polytope of the heap of $\mathsf{sort_c}(w_0)$, the longest c-sorting word of A_{m-1}. Consequently, the normalized volume of $\mathsf{Birk}(c)$ is the number of longest (length $\binom{m}{2}$) chains in Reading's (type A_{m-1}) c-Cambrian lattice, an important generalization of the Tamari lattice [6].

The c-singletons are those permutations whose reduced words appear as order ideals of the heap $\mathsf{Heap}([u])$, where $[u]$ is equal to $\mathsf{sort_c}(w_0)$. In the present paper, we replace $\mathsf{sort_c}(w_0)$ with an arbitrary reduced word $[u]$ (of an arbitrary permutation). That is, given an arbitrary reduced word $[u]$ and its corresponding heap $H = \mathsf{Heap}([u])$, let $\mathsf{Pref}(H)$ denote the set of permutations with a reduced word that is a prefix of a word in the commutation class of $[u]$. We define $\mathsf{Birk}(H)$ to be the convex hull of

$$\{X(w) \mid w \in \mathsf{Pref}(H)\}.$$

Question 1.1 ([1, Question 8.1])**.** Is $\mathsf{Birk}(H)$ integrally equivalent to $\mathcal{O}(H)$?

As we have already observed in [1, Example 8.2], the answer depends on the reduced word $[u]$ and not just on the permutation w. We classify when the answer to Question 1.1 is "yes" in terms of a novel combinatorial model, a collection of digraphs $\Gamma_i(H)$.

Theorem 1.2. Let $[u]$ be a reduced word of a permutation, and let $H = \mathsf{Heap}([u])$. The following are equivalent. (1) $\mathsf{Birk}(H) \cong \mathcal{O}(H)$; (2) $\dim(\mathsf{Birk}(H)) = \dim(\mathcal{O}(H))$; (3) $\Gamma_i(H)$ is a forest for all i.

In [2] we established an analogue of our c-singleton result for type B. The key to the construction of c-Birkhoff polytopes in type B is a natural embedding of the Coxeter group B_n into the Coxeter group A_{2n-1} by identifying each $v \in B_n$ with a permutation $\eta(v) \in S_{2n} = A_{2n-1}$. We defined a (type B) c^B-Birkhoff polytope as the convex hull of permutation matrices $X(\eta(v))$ for all c^B-singletons v in B_n, and proved that $\mathsf{Birk}(c^B)$ is integrally equivalent to the order polytope of the heap of $\mathsf{sort}_{c^B}(w_0^B)$, the longest c^B-sorting word of B_n. The unfolding proof in [2] adapts naturally to the present setting and yields our second main result.

Theorem 1.3. Let $[u]$ be a reduced word in B_n, and let $H = \mathsf{Heap}([u])$. The following are equivalent. (1) $\mathsf{Birk}(H) \cong \mathcal{O}(H)$; (2) $\mathsf{Birk}(\eta(H)) \cong \mathcal{O}(\eta(H))$; (3) $\Gamma_i(\eta(H))$ is a forest for all i.

2 Preliminaries

A *Coxeter system* (W, S) is a *Coxeter group* W together with a set S of generators for W called *simple reflections* subject to the relations $s^2 = e$ for all $s \in S$ and the braid relations

$(st)^{m(s,t)} = e$ for all s, t such that $m(s,t) < \infty$. For $s, t \in S$ where $m(s,t) = 2$, we have $st = ts$, which we call a *commutation relation*. An application of a commutation relation to a product of simple reflections is called a *commutation move*.

Given $w \in W$, the minimum number of simple reflections among all expressions for w as a product of simple reflections is called the *length* of w, and is denoted by $\ell(w)$. A *reduced decomposition* of w is an expression $w = s_{i_1} \cdots s_{i_{\ell(w)}}$ realizing $\ell(w)$.

2.1 Type A_n and B_n permutations

This paper focuses on the Coxeter groups of type A and B, which we denote by A_n and B_n. We now review the combinatorial realizations of these groups in terms of permutations and signed permutations. For more details, see for example [4].

Let A_n denote the symmetric group on $n+1$ elements. The simple reflections for A_n are *adjacent transpositions* $s_k = (k \quad k+1)$, for $k \in [n]$ where $[n] = \{1, 2, \ldots, n\}$. Distinct simple reflections satisfy the commutation relation $s_i s_j = s_j s_i$ if and only if $|i - j| > 1$.

Let B_n be the group of permutations on

$$\pm[n] = \{-n, \ldots, -2, -1, 1, 2, \ldots, n\}$$

which satisfies $w(-k) = -w(k)$ for all $k \in [n]$. The simple reflections for B_n are $s_k^B = (-k-1 \quad -k)(k \quad k+1)$ for $k \in [n-1]$ and $s_0^B = (-1 \quad 1)$. As in A_n, distinct simple reflections in B_n satisfy the commutation relation $s_i^B s_j^B = s_j^B s_i^B$ if and only if $|i - j| > 1$.

To simplify notation, we refer to a reduced decomposition $s_{i_1} \cdots s_{i_{\ell(w)}}$ of w in A_n or B_n via its *reduced word* $\left[i_1 \cdots i_{\ell(w)}\right]$. Given a reduced word $[u]$, the equivalence class consisting of all words that can be obtained from $[u]$ by a sequence of commutation moves is called the *commutation class* of $[u]$.

The groups B_n and A_{2n-1} are related by an "unfolding" homomorphism (see [4, Section 8.1]). Given a simple reflection of B_n, define η to be the map

$$\eta(s_i^B) = \begin{cases} s_n & \text{if } i = 0 \\ s_{n+i} s_{n-i} & \text{if } i > 0. \end{cases}$$

We extend η homomorphically to reduced words in B_n. This map sends reduced words of B_n to reduced words of A_{2n-1}, and all reduced words of $v \in B_n$ map to reduced words of the same element $w \in A_{2n-1}$. Thus by abuse of notation we can write $\eta(v) = w$ considering η as a map from B_n to A_{2n-1}.

2.2 Heaps and prefixes of $[u]$ up to commutation

Let W be the Coxeter group A_n or B_n. We review the classical theory of heaps (see [10] and [8, Solutions to Exercise 3.123(ab)]), following the exposition given in [9].

Definition 2.1. Given a reduced word $[u] = [u_1 \cdots u_\ell]$ of an element in W, consider the partial order $\preccurlyeq$ on the set $\{1, \ldots, \ell\}$ obtained via the transitive closure of the relations $x \prec y$ for $x < y$ such that $|u_x - u_y| \leq 1$. For each $1 \leq x \leq \ell$, the *label* of the poset element x is u_x. This labeled poset is called the *heap* for $[u]$, denoted $\mathsf{Heap}([u])$.

The Hasse diagram for this poset with elements $\{1, \ldots, \ell\}$ replaced by their labels is called the *heap diagram* for $[u]$ or for $\mathsf{Heap}([u])$. In our figures, we represent each label j by the simple reflection s_j for clarity.

Example 2.2. Figures 1 and 2 illustrate the Hasse diagram of the underlying poset of $H = \mathsf{Heap}([u])$ and the heap diagram of H for $[u] = [1213243212]$ and for $[u] = [2123243212]$.

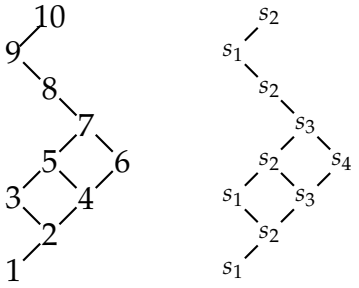


Figure 1: Hasse diagram (left) and heap diagram (right) of the heap of [1213243212].

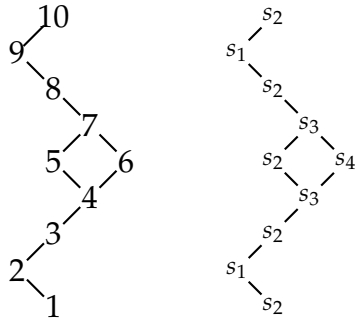


Figure 2: Hasse diagram (left) and heap diagram (right) of the heap of [2123243212].

We can understand the commutation class of $[u]$ through linear extensions of $\mathsf{Heap}([u])$.

Definition 2.3 (labeled linear extension)**.** Let $(H, \preccurlyeq_H)$ be a poset on ℓ elements. A *linear extension* of H is a bijection $\pi : \{1, \ldots, \ell\} \to H$ consistent with the structure of the poset, that is, $\pi(x) \prec_H \pi(y)$ implies $x < y$. For a reduced word $[u] = [u_1 \cdots u_\ell]$, a *labeled linear extension* of $\mathsf{Heap}([u])$ is $\left[u_{\pi(1)} \cdots u_{\pi(\ell)}\right]$ where π is a linear extension of $\mathsf{Heap}([u])$.

Proposition 2.4. [9, Proposition 2.2] Given a reduced word $[u]$, the set of labeled linear extensions of $\mathsf{Heap}([u])$ is the commutation class of $[u]$.

Consider a reduced word $[u] = [u_1 u_2 \ldots u_\ell]$ in W and the labeled poset $H = \mathsf{Heap}([u])$. Given an order ideal I of H, let $[u]_I = [u_i]_{i \in I}$ be the subword of $[u]$ at positions I. Let $\mathsf{Pref}(H)$ denote the set of elements in W having a reduced word which is a prefix of a word in the commutation class of $[u]$. The next lemma follows from Proposition 2.4.

Lemma 2.5. An element $w \in W$ is in $\mathsf{Pref}(H)$ if and only if there exists a (necessarily unique) order ideal I of H such that $[u]_I$ is a reduced word of w.

The *(right) weak order* is a partial order on W defined by $v \leq w$ if some prefix of a reduced word of w is a reduced word of v. We can consider $\mathsf{Pref}(H)$ as a poset with ordering induced by the weak order on W.

Proposition 2.6. Let $\mathrm{perm}\colon J(H) \to \mathsf{Pref}(H)$ be the map that sends I to $w \in W$ that has a reduced word $[u]_I$. The inverse map $f : \mathsf{Pref}(H) \to J(H)$ of perm is $w \mapsto I_w$ where I_w is the order ideal which is guaranteed to exist by Lemma 2.5. Both maps are poset isomporphisms.

2.3 Notation

In this subsection we set notation that will show up frequently throughout this paper. Let $W = A_n$. Fix a reduced word $[u] = [u_1 u_2 \dots u_\ell]$, and let $H = \mathsf{Heap}([u])$. For each element $k \in H = \{1, \dots, \ell\}$, let $\langle k \rangle$ denote the principal order ideal of H generated by k, let $v_k = \mathrm{perm}(\langle k \rangle)$, and let $\widetilde{v}_k = \mathrm{perm}(\langle k \rangle \setminus \{k\})$. Let v_0 denote the identity permutation.

Let $\mathsf{M}_{n+1}(\mathbb{R})$, resp. $\mathsf{M}_{n+1}(\mathbb{Z})$, denote the set of $(n+1) \times (n+1)$ matrices with entries in $\mathbb{R}$, resp. $\mathbb{Z}$. Given $w \in A_n$, let $X(w)$ denote the corresponding permutation matrix in $\mathsf{M}_{n+1}(\mathbb{Z})$. Specifically, entry (i, j) of $X(w)$ is 1 if $w(i) = j$ and is 0 otherwise. Given an order ideal $I \in J(H)$, let $\mathrm{Mat}(I) = X(\mathrm{perm}(I))$. For short, let X_k denote $X(v_k) = \mathrm{Mat}(\langle k \rangle)$ and $\widetilde{X}_k$ denote $X(\widetilde{v}_k) = \mathrm{Mat}(\langle k \rangle \setminus \{k\})$, for each $k \in H$. We will also write X_0 to denote the identity matrix. We will use X to denote an element of $\mathsf{M}_{n+1}(\mathbb{R})$, and $X(i, j)$ will refer to the entry in row i and column j of a matrix X.

Define $\mathsf{Aff}(H)$ to be the affine hull of the vertices of $\mathsf{Birk}(H)$. Since $\mathsf{Aff}(H)$ is an affine subspace of the affine hull of the Birkhoff polytope, we have the following row and column relations satisfied by points by every point X in $\mathsf{Aff}(H)$:

$$\text{for all } 1 \leq k \leq n+1, \qquad \sum_{j=1}^{n+1} X(j,k) = 1 \qquad \text{and} \qquad \sum_{j=1}^{n+1} X(k,j) = 1. \tag{2.1}$$

3 Gamma graphs

In this section we will introduce the notion of *i-classes*, which can be used to calculate additional linear relations satisfied by points in $\mathsf{Aff}(H)$.

Definition 3.1 (Γ_i digraph and i-classes)**.** Let H be the heap for the reduced word $[u]$ in A_n. We build one digraph $\Gamma_i(H)$ for every $1 \leq i \leq n$ as follows. For each i, we set the vertex set $V(\Gamma_i(H)) = \{1, 2, \dots, n+1\}$. For each element $j \in H$ with label i, we add a directed arrow from $v_j(i+1)$ to $v_j(i)$. We label this arrow with j.

Given $1 \leq i \leq n$, the *i-classes* of H are the blocks of the partition of $[n+1]$ arising from the connected components of $\Gamma_i(H)$.

Example 3.2. Let $[u] = [1213243212]$ and let H denote its heap (see Figure 1). The digraphs $\Gamma_1(H), \Gamma_2(H), \Gamma_3(H)$, and $\Gamma_4(H)$ are drawn in Figure 3.

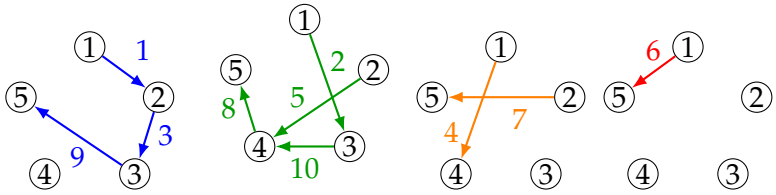


Figure 3: From left to right: Graphs $\Gamma_1(H)$, $\Gamma_2(H)$, $\Gamma_3(H)$, and $\Gamma_4(H)$ for $H = \mathsf{Heap}([121324321])$

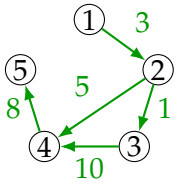


Figure 4: Graph $\Gamma_2(H)$ for $H = \mathsf{Heap}([2123243212])$.

Proposition 3.3 (Γ_i Relations)**.** Let C be an i-class of H. If $X \in \mathsf{Aff}(H)$, then

$$\sum_{j=1}^{i} \sum_{y \in C} X(j, y) = |C \cap \{1, 2, \ldots, i\}|.$$

Proof Sketch. It suffices to show the statement for the matrices $\mathrm{Mat}(I)$ for $I \in J(H)$. This follows from a straightforward argument inducting on the number of instances of i in $[u]$. The most interesting step is when C contains the last edge added to $\Gamma_i(H)$. Either this edge combined two connected components of a prefix of $[u]$ or C was already a connected component, and in either case the inductive hypothesis finishes the proof. □

One consequence of our main result, Theorem 1.2, is that if each $\Gamma_i(H)$ is a forest, then the relations in Proposition 3.3 along with the row and column relations (Equation (2.1)) completely describe $\mathsf{Aff}(H)$.

Proposition 3.4. Suppose there exists an (undirected) cycle C in $\Gamma_i(H)$. Choose a planar embedding of C. If $a_1, \ldots, a_r$ are the clockwise edges and $b_1, \ldots, b_s$ are the counterclockwise edges in this embedding, then

$$\sum_{j=1}^{r} \left(X_{a_j} - \widetilde{X}_{a_j} \right) - \sum_{j=1}^{s} \left(X_{b_j} - \widetilde{X}_{b_j} \right) = \mathbf{0}. \tag{3.1}$$

In particular, if a is a label of an edge on this cycle, then we can write X_a as an affine combination of matrices $\{X_k \mid k \neq a\}$.

Example 3.5. Let $[u] = [2123243212]$ and let H be its heap. The graph $\Gamma_2(H)$ (see Figure 4) has an undirected 3-cycle. From this 3-cycle, by (3.1) we have the relation $X_{10} - \widetilde{X}_{10} + X_1 - \widetilde{X}_1 - X_5 + \widetilde{X}_5 = \mathbf{0}$. Each $\widetilde{v}_a$ which appears is in fact v_{a-1}, so we can rewrite this relation $X_{10} = X_9 - X_1 + X_0 + X_5 - X_4$. That is, X_{10} is in the affine hull of $\{X_0, \ldots, X_9\}$ as promised by Proposition 3.4.

Lemma 3.6 (Inclusion-exclusion relation)**.** Let $I, K \in J(H)$. We have

$$\mathrm{Mat}(I \cup K) = \mathrm{Mat}(I) + \mathrm{Mat}(K) - \mathrm{Mat}(I \cap K).$$

The following result is a consequence of this inclusion-exclusion relation and the uniqueness of upper-bounds property of H.

Proposition 3.7. For every $I \in J(H)$, the matrix $\mathrm{Mat}(I)$ can be written as an integer affine combination of $X_0, \ldots, X_{|H|}$. In particular, $\mathsf{Aff}(H)$ is equal to the affine hull of $\{X_0, \ldots, X_{|H|}\}$ and the dimension of $\mathsf{Birk}(H)$ is at most $|H|$.

The following is an immediate consequence of Proposition 3.4 and Proposition 3.7.

Theorem 3.8. If $\Gamma_i(H)$ contains an (undirected) cycle for some i, then $\dim(\mathsf{Birk}(H)) < \dim(\mathcal{O}(H))$.

4 An affine projection

Throughout this section let H be a heap of a reduced word $[u]$ of length $\ell \geq 1$ in A_n. Given a nonnegative integer $a \leq \ell$, let $[u]_a$ denote the length a prefix of u, and let H_a denote the heap of $[u]_a$.

Definition 4.1 (Projection Π_H via affine functionals π_k)**.** Suppose that $\Gamma_i(H)$ is a forest for all i. We define a projection Π_H from $\mathsf{Aff}(H)$ to $\mathbb{R}^\ell$ using a collection of affine functionals.

For each $k \in H = \{1, \ldots, \ell\}$, we will construct the affine functional $\pi_k : \mathsf{M}_{n+1}(\mathbb{R}) \to \mathbb{R}$ as follows. Let i be the label of k, and let Y_k and Z_k to be two i-classes of H_{k-1} such that $\min Y_k < \min Z_k$ and $Y_k \sqcup Z_k$ is an i-class of H_k. Note that since $\Gamma_i(H)$ forest, Y_k, Z_k exist (and are nonempty). Let m_k denote $|Y_k \cap \{1, 2, \ldots, i\}|$ and define

$$\pi_k(X) = m_k - \sum_{s=1}^{i} \sum_{y \in Y_k} X(s, y). \tag{4.1}$$

Finally, we define $\Pi_H : \mathsf{Aff}(H) \to \mathbb{R}^\ell$ to be the affine map that sends X to the vector that has kth entry $\pi_k(X)$.

Example 4.2. Consider the reduced word $[u] = [1213243212]$ and its heap H (see Figure 1). The computation for the affine functionals (4.1) is given in Table 1.

Let V be the $\ell \times \ell$ integer matrix with $\Pi_H(X_j)$ in column j, that is, $V(k, j) = \pi_k(X_j)$.

Theorem 4.3. If $\Gamma_i(H)$ is a forest for all i, then V is an upper triangular matrix with ± 1 for all diagonal entries.

k	i	v_k	i-classes of H_{k-1}	i-classes of H_k	Y_k	m_k	$\pi_k(X)$
1	1	21345	1,2,3,4,5	12,3,4,5	1	1	$1 - X(1,1)$
2	2	23145	1,2,3,4,5	13,2,4,5	1	1	$1 - X(1,1) - X(2,1)$
3	1	32145	12,3,4,5	123,4,5	12	1	$1 - X(1,1) - X(1,2)$
4	3	23415	1,2,3,4,5	14,2,3,5	1	1	$1 - X(1,1) - X(2,1) - X(3,1)$
5	2	34215	13,2,4,5	13,24,5	2	1	$1 - X(1,2) - X(2,2)$
6	4	23451	1,2,3,4,5	15,2,3,4	1	1	$1 - X(1,1) - X(2,1) - X(3,1) - X(4,1)$
7	3	34521	14,2,3,5	14,25,3	2	1	$1 - X(1,2) - X(2,2) - X(3,2)$
8	2	35421	13,24,5	13,245	24	1	$1 - X(1,2) - X(2,2) - X(1,4) - X(2,4)$
9	1	53421	123,4,5	1235,4	123	1	$1 - X(1,1) - X(1,2) - X(1,3)$
10	2	54321	13,245	12345	13	1	$1 - X(1,1) - X(2,1) - X(1,3) - X(2,3)$

Table 1: Computation for the affine functionals of Example 4.2

Proof sketch. Since Y_k is an i-class of H_{k-1}, we have $\pi_k(X) = m_k - \sum_{s=1}^{i} \sum_{y \in Y_k} X(s,y) = 0$, for all $X \in \mathsf{Aff}(H_{k-1})$ by Proposition 3.3. Using this, we can deduce that $\pi_k(X_1) = \cdots = \pi_k(X_{k-1}) = 0$ and $\pi_k(X_k) = \pm 1$ for each k. □

Theorem 4.4. Suppose that $\Gamma_i(H)$ is a forest for all i. The affine map Π_H is a injection that maps lattice points to lattice points.

Proof. Let $X \in \mathsf{Aff}(H)$. Since the functional π_k is built from the i-classes of H_k, each value $\pi_k(X)$ allows us to uniquely solve for one new entry of X using the entries in the first $i-1$ rows and already determined entries of row i. This process recovers the first n rows of X, and the $(n+1)$-st row is determined from the first n rows with the column relations in (2.1). Therefore $\Pi_H(X) = \Pi_H(X')$ implies $X = X'$, i.e., Π_H is injective. From the definition of π_k, it is clear that if X is integral then $\pi_k(X)$ is integral for all k, and therefore $\Pi_H(X)$ is integral. □

5 Proof of main result in type A

Let $[u]$ be a reduced word of length $\ell \geq 1$ in A_n with heap H. Given $I \in J(H)$, let $o(I)$ be the indicator vector $o(I) = (o_1, \ldots, o_\ell) \in \mathbb{R}^\ell$ where $o_i = 1$ if $i \in I$ and otherwise $o_i = 0$.

Recall $\mathcal{O}(H)$ denotes the order polytope of H, whose vertices are the indicator vectors $o(I)$ for all order ideals I of H. $\mathcal{O}(H)$ is full-dimensional, implying the affine hull of $\mathcal{O}(H)$ is the entire $\mathbb{R}^\ell$ and the set of lattice points in this affine hull is $\mathbb{Z}^\ell$.

Let O denote the $\ell \times \ell$ matrix with $o(\langle k \rangle)$ in column k. Note that O is an upper triangular, integer matrix with $O(k,k) = 1$, so $\det(O) = 1$. Furthermore, its inverse O^{-1} is an integer matrix with determinant 1.

Theorem 5.1. Suppose $\Gamma_i(H)$ is a forest for all i.

1. Let $U = OV^{-1}$. Then U is an integer matrix with $\det(U) = \pm 1$ such that $UV = O$. In particular, U is a bijection from $\mathbb{R}^\ell$ onto itself, and U restricts to a bijection from $\mathbb{Z}^\ell$ onto itself.

2. For all $I \in J(H)$, we have $U \circ \Pi_H(\mathrm{Mat}(I)) = o(I)$.

3. Let $T : \mathsf{Aff}(H) \to \mathbb{R}^\ell$ be defined by $T = U \circ \Pi_H$. Then T is an affine bijection that restricts to bijections $\mathsf{Birk}(H) \to \mathcal{O}(H)$ and $\mathsf{Aff}(H) \cap \mathsf{M}_{n+1}(\mathbb{Z}) \to \mathbb{Z}^\ell$.

Proof. (1) It follows from Theorem 4.3 that V^{-1} is an integer matrices with determinant ± 1. This is also true of O by definition. So $U = OV^{-1}$ is an integer matrix with determinant ± 1. This fact also implies the second statement.

(2) For the principal order ideals $\langle k \rangle$, this is by construction. For other order ideals, this follows from the fact that for all $I, K \in J(H)$ we have $o(I \cup K) = o(I) + o(K) - o(I \cap K)$, which matches the relation in Lemma 3.6.

(3) T is an affine map because it is the composition of two affine maps. Π_H is injective by Theorem 4.4 and $\mathcal{U}$ is injective by (1), so T is injective. By (2), all vertices of $\mathcal{O}(H)$ are in the image of T. Since T is affine, T is surjective onto the affine hull of $\mathcal{O}(H)$, which is all of $\mathbb{R}^\ell$, and from $\mathsf{Birk}(H)$ onto $\mathcal{O}(H)$.

Π_H and $\mathcal{U}$ map lattice points to lattice points by Theorem 4.4 and (1), respectively, so T does as well. Suppose $\mathbf{z} \in \mathbb{Z}^\ell$. Let $\mathbf{c} = O^{-1}\mathbf{z}$. Since O^{-1} is an integer matrix, $\mathbf{c} = (c_1, \ldots, c_\ell) \in \mathbb{Z}^\ell$. Let $c_0 = 1 - \sum_{k=1}^{\ell} c_k$. Then $\mathbf{z} = O\mathbf{c} = \sum_{k=1}^{\ell} c_k o(\langle k \rangle) = c_0\mathbf{0} + \sum_{k=1}^{\ell} c_k o(\langle k \rangle)$. Since $\mathbf{0}$ is the indicator vector of the empty order ideal and T is affine, $T^{-1}(\mathbf{z}) = \sum_{k=0}^{\ell} c_k X_k$, a lattice point in $\mathsf{Aff}(H)$. So, T^{-1} maps lattice points to lattice points and T restricts to a bijection from the lattice points of $\mathsf{Aff}(H)$ onto $\mathbb{Z}^\ell$. □

Proof of Theorem 1.2. Clearly, (1) implies (2). Theorem 5.1 tells us that (3) implies (1), and we have (2) implies (3) by Theorem 3.8. □

6 Proof of main result in type B

Recall from Section 2.1 that the map η sends elements of B_n to elements of A_{2n-1}. We can use this map to define $\mathsf{Birk}(H)$ when H is the heap of a type B reduced word.

Definition 6.1. Let $[u]$ be a reduced word in B_n and $H = \mathsf{Heap}([u])$. Then define $\mathsf{Birk}(H)$ to be the convex hull of

$$\{X(\eta(w)) \mid w \in \mathsf{Pref}(H)\}.$$

As in the type A case, define $\mathsf{Aff}(H)$ to be the affine hull of the vertices of $\mathsf{Birk}(H)$.

For the rest of this paper, let $[u]$ be a nonempty reduced word in B_n and $H = \mathsf{Heap}([u])$. Let $\eta(H)$ denote $\mathsf{Heap}(\eta([u]))$. For $I \in J(H)$, we define $\eta(I) = \mathsf{Heap}(\eta([u]_I))$. Since $\eta([u]_I)$ is a prefix of $\eta([u])$ up to commutation, $\eta(I)$ can be viewed as an order ideal of $\eta(H)$. We have following useful facts about $\eta(H)$.

Proposition 6.2. Let $[u]$ be a reduced word in B_n and $H = \mathsf{Heap}([u])$.

1. $\eta(H)$ has reflectional symmetry with respect to the vertical axis. The right side has the same underlying poset as H and the elements are numbered in the same order. Similarly, the left side has the same underlying poset as a flip of H across a vertical axis and the elements are numbered in the same order.

2. If $w \in \mathsf{Pref}(H)$ then $\eta(w) \in \mathsf{Pref}(\eta(H))$.

Part (1) of the above proposition follows from the construction of the map η. Part (2) is analogous to Lemma 3.6 from [2] and the proof is similar. It follows from this statement that $\mathsf{Birk}(H)$ is a subset of $\mathsf{Birk}(\eta(H))$ and $\mathsf{Aff}(H)$ is a subset of $\mathsf{Aff}(\eta(H))$.

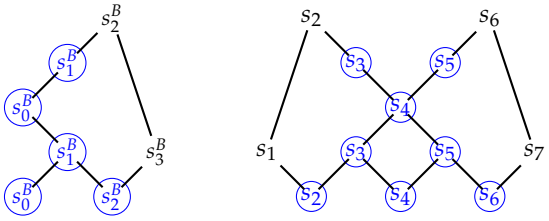


Figure 5: For $[u] = [0213012]$, the heap diagram of H with elements of an order ideal I circled (left) and that of $\eta(H)$ with $\eta(I)$ circled (right).

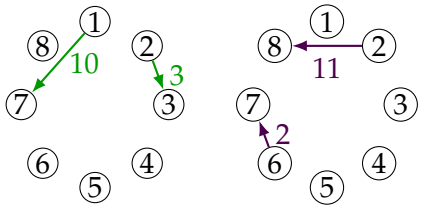


Figure 6: $\Gamma_2(\eta(H))$ (left) and $\Gamma_6(\eta(H))$ (right) for $[u] = [0213012]$.

Lemma 6.3. There is an edge from x to y in $\Gamma_i(\eta(H))$ exactly when there is an edge from $2n+1-y$ to $2n+1-x$ in $\Gamma_{2n-i}(\eta(H))$.

Example 6.4. If $[u] = [0213012] \in B_4$, then $\eta([u]) = [462537145362] \in A_7$. Figure 5 depicts the heap diagrams for H and $\eta(H)$. It also shows an order ideal I and $\eta(I)$. Figure 6 illustrates Lemma 6.3.

Lemma 6.5. The image of B_n under η is the set of elements of A_{2n-1} with permutation matrices that are invariant under 180 degree rotation.

The above lemma implies that elements of $\mathsf{Aff}(H)$ satisfy the equations in (2.1) and Proposition 3.3 as well as being invariant under 180 degree rotation, that is, for $X \in \mathsf{Aff}(H)$, $X(i,j) = X(2n+1-i, 2n+1-j)$. In addition, Lemma 6.3 tells us that the $\Gamma_i(\eta(H))$ relations from Proposition 3.3 and the 180 degree rotation invariance together imply the $\Gamma_{2n-i}(\eta(H))$ relations from Proposition 3.3. Thus, $\mathsf{Aff}(H)$ is a subset of the elements of $\mathsf{M}_{2n}(\mathbb{R})$ that satisfy the equations in (2.1), satisfy the equation in Proposition 3.3 for $i \leq n$, and are 180 degree rotation invariant. We are now ready to define the projection Π_H in type B.

Definition 6.6. Suppose that $\Gamma_i(\eta(H))$ is a forest for all i. For each k on the left side of $\eta(H)$, that is, $k \in \eta(H) = \{1, \ldots, |\eta(H)|\}$ with label at most n, construct the affine functional π_k as in Definition 4.1. Note that by Proposition 6.2(1) this produces $|H|$ functions. Then define $\Pi_H : \mathsf{Aff}(H) \to \mathbb{R}^{|H|}$ to be the affine map that sends X to the vector that has entries $\pi_k(X)$, in order of the index k, for each k on the left side of $\eta(H)$.

Example 6.7. Continuing Example 6.4, Π_H sends $X \in \mathsf{Aff}(H)$ to the vector

$$(\pi_1(X), \pi_3(X), \pi_5(X), \pi_7(X), \pi_8(X), \pi_{10}(X), \pi_{12}(X)) \in \mathbb{R}^7 = \mathbb{R}^{|H|}$$

where the functionals π_k are defined using $\eta(H)$.

Theorem 6.8. Suppose that $\Gamma_i(\eta(H))$ is a forest for all i. The map $\Pi_H : \mathsf{Aff}(H) \to \mathbb{R}^{|H|}$ is injective and maps lattice points to lattice points.

Proof. Suppose $X \in \mathsf{Aff}(H)$. Using the same process described in Theorem 4.4, we can determine the first n rows of X from $\Pi_H(X)$. Since all matrices in $\mathsf{Aff}(H)$ have 180 degree rotational symmetry, we can use this to determine the last n rows of X. This shows Π_H is injective. From the definition of π_k, we can that if X is integral then $\pi_k(X)$ is integral for all k, and therefore $\Pi_H(X)$ is integral. □

For H where $\Gamma_i(\eta(H))$ is a forest for all i we have the following commutative diagram.

$$\begin{array}{ccccc} \mathsf{Aff}(\eta(H)) & \xrightarrow{\Pi_{\eta(H)}} & \Pi_{\eta(H)}(\mathsf{Aff}(\eta(H))) & \xrightarrow{\mathcal{U}_{\eta(H)}} & \mathbb{R}^{|\eta(H)|} \\ \uparrow \iota & & & & \downarrow P_H \\ \mathsf{Aff}(H) & \xrightarrow{\Pi_H} & \Pi_H(\mathsf{Aff}(H)) & \dashrightarrow^{\mathcal{U}_H} & \mathbb{R}^{|H|} \end{array} \tag{6.1}$$

In the diagram (6.1) above, ι is the inclusion map, $\Pi_{\eta(H)}$ is the type A map from Definition 4.1, Π_H is the type B map from Definition 6.6, and $\mathcal{U}_{\eta(H)}$ is multiplication by the matrix $\mathcal{U}$ from Theorem 5.1. The map P_H acts on elements of $\mathbb{R}^{|\eta(H)|}$ by deleting coordinate k whenever k has a label greater that n as an element of $\eta(H)$; note that Proposition 6.2(1) tells us this process gives us an element of $\mathbb{R}^{|H|}$. Finally, we define $\mathcal{U}_H$ as the composition of maps that makes the diagram commute. Since Π_H is injective (Theorem 6.8), it is bijective if we restrict its codomain to $\Pi_H(\mathsf{Aff}(H))$. Thus it has an inverse $\Pi_H^{-1} : \Pi_H(\mathsf{Aff}(H)) \to \mathsf{Aff}(H)$. We therefore define $\mathcal{U}_H : \Pi_H(\mathsf{Aff}(H)) \to \mathbb{R}^{|H|}$ as

$$\mathcal{U}_H = P_H \circ \mathcal{U}_{\eta(H)} \circ \Pi_{\eta(H)} \circ \iota \circ \Pi_H^{-1}.$$

Theorem 6.9. Suppose $\Gamma_i(H)$ is a forest for all i. Then the following are true.

1. If I is an order ideal of H then $\mathcal{U}_H \circ \Pi_H(\mathrm{Mat}(\eta(I))) = o(I)$.

2. $\mathcal{U}_H \circ \Pi_H$ is a bijection that restricts to a bijection between $\mathsf{Birk}(H)$ and $\mathcal{O}(H)$ and also restricts to a bijection between lattice points in the domain and lattice points in the codomain.

Proof sketch. We can prove this theorem using properties of $\mathcal{U}_{\eta(H)} \circ \Pi_{\eta(H)}$ from Theorem 5.1, the definitions of ι and P_H, and the fact that $\mathcal{U}_H \circ \Pi_H = P_H \circ \mathcal{U}_{\eta(H)} \circ \Pi_{\eta(H)} \circ \iota$. □

Proof sketch of Theorem 1.3. We have (2) and (3) are equivalent by Theorem 1.2. We have (3) implies (1) by Theorem 6.9 (2), and the converse is analogous to the type A case. □

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AI disclosure

- about exposition:
  - We used an AI tool (Claude Sonnet 5.5) for grammar checks and language tightening.
- about figures:
  - Our figures were created using TikZ and SageMath. We used an AI tool (ChatGPT 5.6 Sol) to help us edit the TikZ code.
- about references:
  - We used AI tools (such as Claude Sonnet 5.5, Opus 5.5, Fable 5.1, and Google AI Overview) to search for literature references.