

# Cluster algebras and binary words

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# Outline

- ▶ Bijection between binary subwords and order filters of a poset
- ▶ Bijection between binary subwords and perfect matchings of a snake graph (terms of a cluster variable)

# Binary word

## Definition

- ▶ A *binary word* is a finite sequence of letters belonging to  $\{0, 1\}$ . **In this talk, consider only words that start with 1.**
  - ▶ Example: 10100.
- ▶ A *subword* is a subsequence of a word.
  - ▶ Example: Some subwords of 10100 are the empty word, 11, 100, and itself.
  - ▶ Non-examples: 10001, 11000 are *not* subwords of 10100.
  - ▶ Note: Even though 1010 appears twice as a subsequence of 10100, we treat it as one subword.

# Subwords of 1011101100

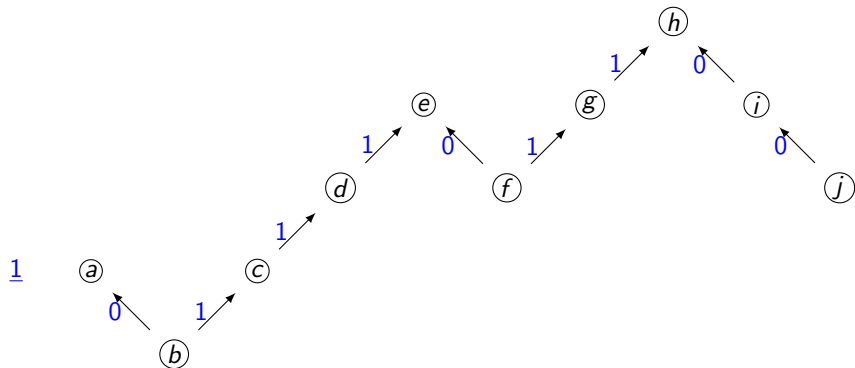
|     |       |     |        |     |         |     |            |
|-----|-------|-----|--------|-----|---------|-----|------------|
| 1.  | empty | 22. | 10110  | 43. | 111010  | 64. | 1111011    |
| 2.  | 1     | 23. | 10111  | 44. | 111011  | 65. | 1111100    |
| 3.  | 10    | 24. | 11000  | 45. | 111100  | 66. | 1111110    |
| 4.  | 11    | 25. | 11010  | 46. | 111101  | 67. | 10101100   |
| 5.  | 100   | 26. | 11011  | 47. | 111110  | 68. | 10110100   |
| 6.  | 101   | 27. | 11100  | 48. | 111111  | 69. | 10110110   |
| 7.  | 110   | 28. | 11101  | 49. | 1001100 | 70. | 10111000   |
| 8.  | 111   | 29. | 11110  | 50. | 1010100 | 71. | 10111010   |
| 9.  | 1000  | 30. | 11111  | 51. | 1010110 | 72. | 10111011   |
| 10. | 1001  | 31. | 100100 | 52. | 1011000 | 73. | 10111100   |
| 11. | 1010  | 32. | 100110 | 53. | 1011010 | 74. | 10111110   |
| 12. | 1011  | 33. | 101000 | 54. | 1011011 | 75. | 11101100   |
| 13. | 1100  | 34. | 101010 | 55. | 1011100 | 76. | 11110100   |
| 14. | 1101  | 35. | 101011 | 56. | 1011101 | 77. | 11110110   |
| 15. | 1110  | 36. | 101100 | 57. | 1011110 | 78. | 11111100   |
| 16. | 1111  | 37. | 101101 | 58. | 1011111 | 79. | 101101100  |
| 17. | 10000 | 38. | 101110 | 59. | 1101100 | 80. | 101110100  |
| 18. | 10010 | 39. | 101111 | 60. | 1110100 | 81. | 101110110  |
| 19. | 10011 | 40. | 110100 | 61. | 1110110 | 82. | 101111100  |
| 20. | 10100 | 41. | 110110 | 62. | 1111000 | 83. | 111101100  |
| 21. | 10101 | 42. | 111000 | 63. | 1111010 | 84. | 1011101100 |

## Binary word to poset

Associate a subword  $w = w_1 w_2 \dots w_n$  to the Hasse diagram of a “line” poset with  $n$  elements  $V_1, V_2, \dots, V_n$  by assigning each  $w_i$  ( $i \geq 2$ ) so that

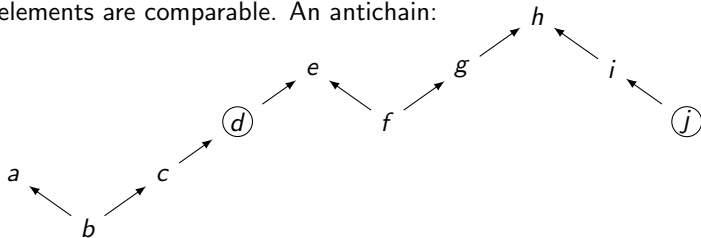
1 corresponds to  $V_{i-1} \nearrow V_i$ , and 0 corresponds to  $V_{i-1} \nwarrow V_i$ .

Example:  $w = 1011101100$

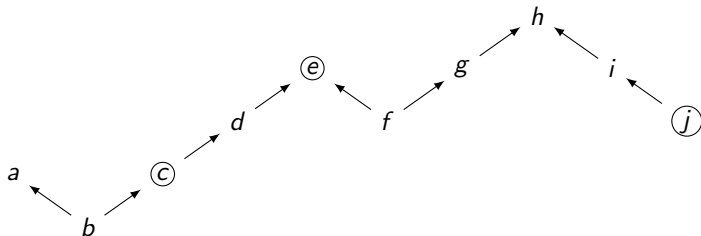


# Antichain

An *antichain* of a poset  $P$  is a subset of  $P$  such that no 2 distinct elements are comparable. An antichain:



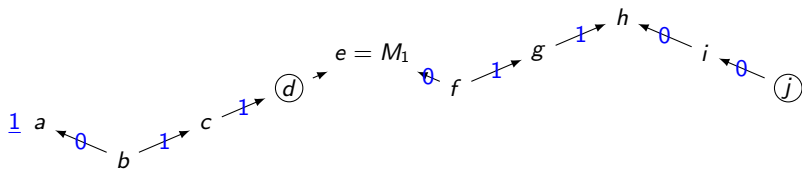
Not an antichain:



## Bijection from antichains $A$ to subwords $s$ ( $G$ .)

The empty antichain is mapped to the empty word. Otherwise, map the antichain  $A = \{A_1, A_2, \dots, A_r\}$  to the following subword of  $w$ :

- ▶ 1 is the first letter.  $s = 1$ \_\_\_\_\_
- ▶ The next letters are the (possibly empty) sequence of edge labels between the first element of  $P$  and  $A_1$ .  $s = 1$  011\_\_\_\_\_



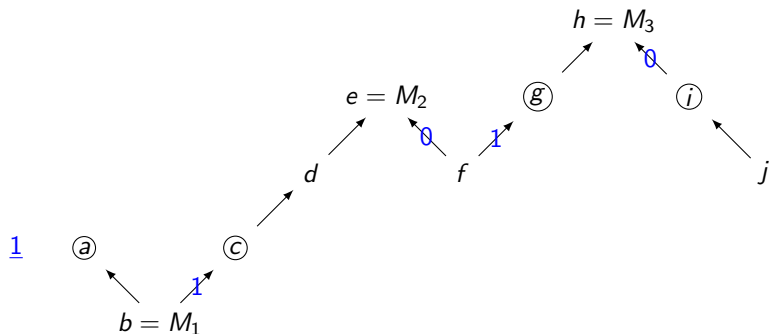
- ▶ If  $A$  contains only one element, we are done. Otherwise, jump to the first element  $M_1$  appearing after  $A_1$  which is either minimal or maximal. The elements of  $P$  between  $A_1$  and  $M_1$  (inclusive) are all comparable to  $A_1$ . Since  $A$  is an antichain, none of these are in  $A$ .
- ▶ Record the labels of the edges between  $M_1$  and  $A_2$ .  $s = 1011$  01100
- ▶ Jump to the first element  $M_2$  appearing after  $A_2$  which is either minimal or maximal. Record the labels of the edges between  $M_2$  and  $A_3$ , and so on.

# Bijection from antichains $A$ to subwords $s$ (G.)

Another example:

The antichain  $A = \{A_1 = \textcircled{a}, A_2 = \textcircled{c}, A_3 = \textcircled{g}, A_4 = \textcircled{i}\}$  of  $P$

is mapped to the subword  $s = \underline{1} \square \underline{1} \square \underline{01} \square \underline{0}$  of  $w$ .



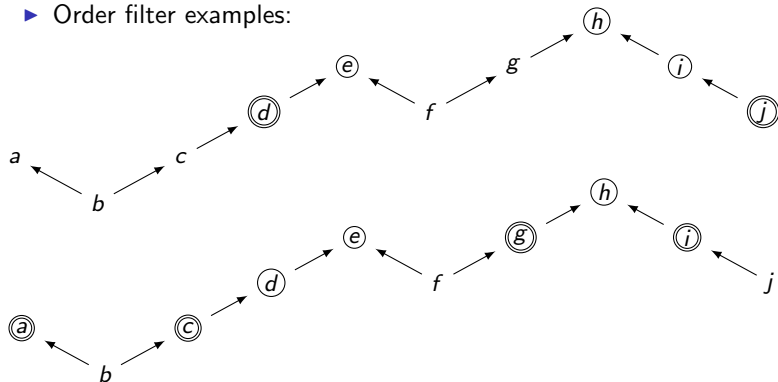


# Order filter

## Definition

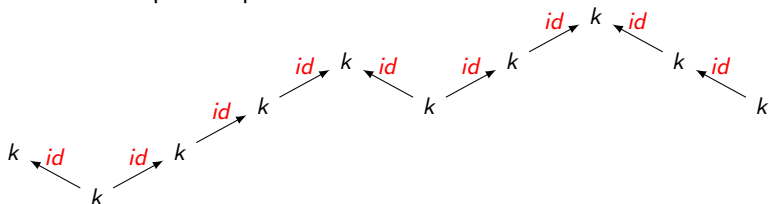
An *order filter* (or dual order ideal) is a subset  $F$  of  $P$  such that if  $t \in F$  and  $s \geq t$ , then  $s \in F$ .

- ▶ Fact: There is a one-to-one correspondence between antichains  $A$  and order filters  $F$ , where  $A$  is the set of minimal elements of  $F$ .
- ▶ Order filter examples:



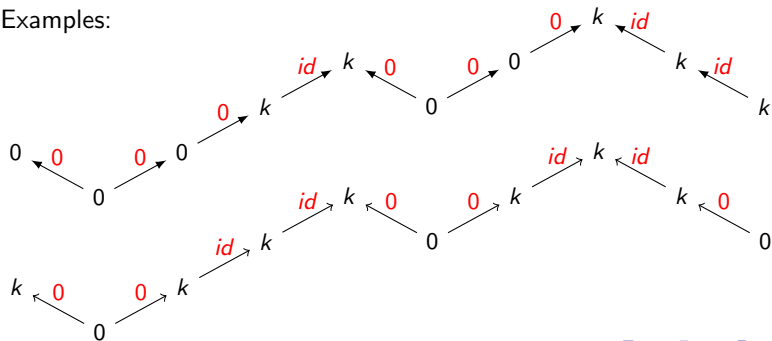
# Why order filter (as opposed to order ideal)?

Consider the quiver representation  $M$  over a field  $k$



The subrepresentations of  $M$  correspond to the order filters of the poset.

Examples:

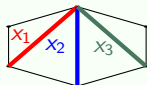


# Cluster algebras (Fomin and Zelevinsky, 2000)

A **cluster algebra** is a subring of  $\mathbb{Q}(x_1, \dots, x_n)$  with a distinguished set of generators, called **cluster variables**.

## Cluster algebras from surfaces (Fomin, Shapiro, and Thurston, 2006, etc.)

- ▶ A Riemann surface  $S$  + marked points gives rise to a cluster algebra.
- ▶ Starting from a triangulation and initial cluster variables  $x_1, \dots, x_n$ , produce all the other cluster variables by an iterative process called



mutation.

- ▶ The cluster variables  $\longleftrightarrow$  curves between marked points, called arcs.


**Laurent Phenomenon** (Fomin - Zelevinsky) and **positivity** (Lee - Schiffler, Gross - Hacking - Keel - Kontsevich, 2014, and special cases by others): Each cluster variable can be expressed as a Laurent polynomial in  $\{x_1, \dots, x_n\}$ , that is, as

$$\frac{f(x_1, \dots, x_n)}{x_1^{d_1} \dots x_n^{d_n}},$$

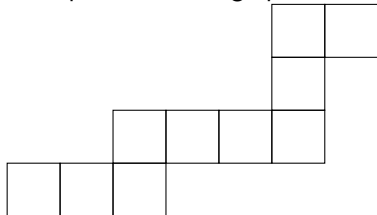
where  $f$  is a polynomial with positive coefficients.

# Snake graphs

## Definition

A *snake graph* is a connected sequence of square tiles . To build a snake graph, start with one tile, then glue a new tile to the north or the east of the previous tile.

Example of a snake graph with 10 tiles:



- ▶ History: Used by Musiker, Propp, Schiffler, and Williams to study positivity and bases of cluster algebras from surfaces (2005, 2009–10). The theory of abstract snake graph was developed further by Çanakçı, Lee, and Schiffler (2012–17).

# Sign function

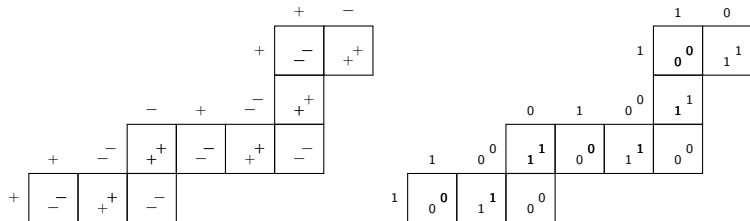
## Definition (Çanakçı, Schiffler)

A *sign function* on a snake graph  $G$  is a map from the set of edges of  $G$  to  $\{+, -\}$  such that, for every tile of  $G$ ,

- ▶ the north and the west edges have the same sign,
- ▶ the south and the east edges have the same sign, and

$$\begin{array}{c}
 + \\
 + \square - \\
 -
 \end{array}
 \quad
 \begin{array}{c}
 - \\
 - \square + \\
 +
 \end{array}$$

- ▶ the sign on the north edge is opposite to the sign on the south edge.

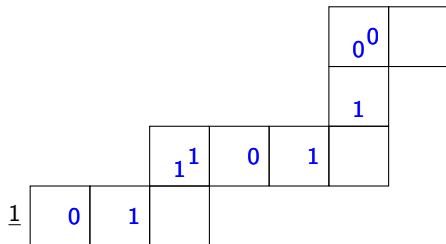


Note: There are two possible sign functions on  $G$ .

For convenience, we replace  $+$  with  $1$  and  $-$  with  $0$ .

# Sign Sequence

The sequence of signs of the interior edges corresponds to a binary word.

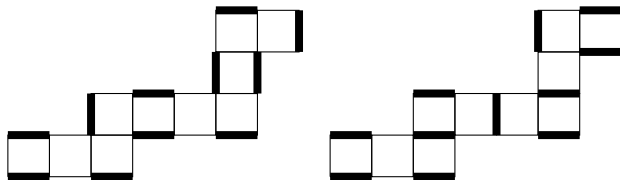


Example: the word 1011101100.

# Perfect matchings

## Definition

A *matching* of a graph  $G$  is a subset of non-adjacent edges of the graph. A *perfect matching* of  $G$  is a matching where every vertex of  $G$  is adjacent to exactly one edge of the matching.



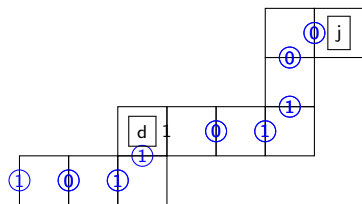
## Theorem ( Musiker, Schiffler, and Williams, 2009–10)

*Each cluster variable (of a cluster algebra from a surface) can be written as the sum of the weights of all perfect matchings of a certain snake graph.*

*(Note: this demonstrates that the Laurent polynomial expansion has all positive coefficients).*

# Bijection from subwords to perfect matchings (G.)

- ▶ Highlight the internal edges of  $G$  corresponding to  $s$ .  
 Example:  $w = 1011101100$  and subword  $s = 101101100$ .  
 Highlight the edges 1011101100 below.
- ▶ For each path  $L$  of consecutive highlighted edges, let  $\square_L$  be the tile which is north/east of the last edge in  $L$ .

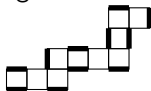


$$\square_{1011} = \boxed{d} \text{ and } \square_{01100} = \boxed{j}$$



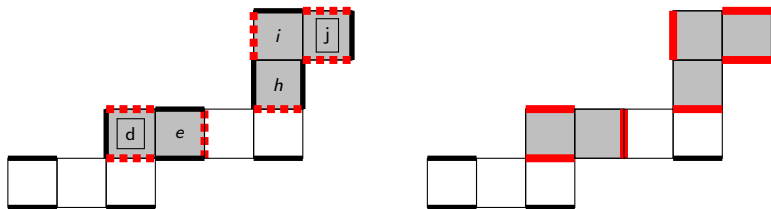
# Bijection from subwords to perfect matchings (G.)

- ▶ Let  $P_{\min}$  be the perfect matching which contains the first south edge

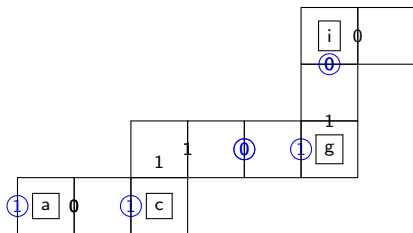


and only boundary edges.

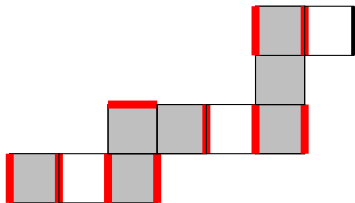
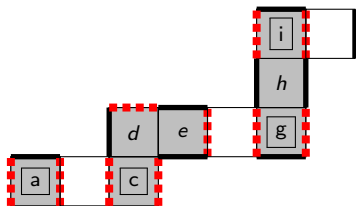
- ▶ Let  $fil(\square_L)$  be the minimal connected sequence of tiles such that  $\square_L \in fil(\square_L)$  and the edges bounding  $fil(\square_L)$  not in  $P_{\min}$  forms a perfect matching of  $fil(\square_L)$ .
- ▶ Let  $fil(s) = \bigcup_L fil(\square_L)$ . Let  $pm(s) = \{\text{edges bounding } fil(s)\} \ominus P_{\min}$ .



# Another example: From 11010 to a perfect matching



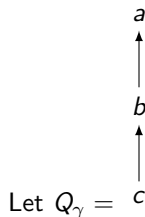
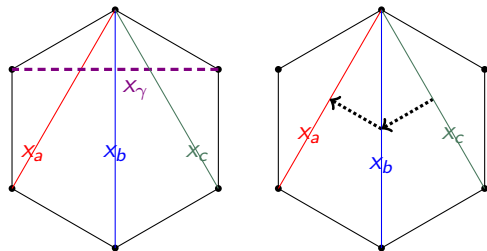
$$\square_1 = a, \square_1 = c, \square_{01} = g, \text{ and } \square_0 = i.$$



# Thank you

Comments and suggestions are welcome

## Example: arc to poset



Motivation:

**Theorem ( Musiker, Schiffler, and Williams, 2011)**

*The order filters of  $Q_\gamma$  are in bijection with the terms of the cluster variable expansion of  $x_\gamma$  with respect to  $x_a$ ,  $x_b$ , and  $x_c$  (where the set of elements of each order filter corresponds to a term in the  $F$ -polynomial).*